HONG KONG STATISTICAL SOCIETY

2015 EXAMINATIONS – SOLUTIONS

GRADUATE DIPLOMA – MODULE 3

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1(a) $p_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$ 1.

$p_{ij}^{(n+m)} = P(X_{n+m} = j \mid X_0 = i)$

$= \sum_k P(X_{n+m} = j, X_n = k \mid X_0 = i)$ 1.

$= \sum_k P(X_{n+m} = j \mid X_n = k, X_0 = i) P(X_n = k \mid X_0 = i)$

$= \sum_k P(X_{n+m} = j \mid X_n = k) P(X_n = k \mid X_0 = i)$, using the Markov property 1.

/5 $= \sum_k p_{kj}^{(m)} p_{ik}^{(n)}$, using time homogeneity 1.

$= \sum_k p_{ik}^{(n)} p_{kj}^{(m)}$ 1.

b(i) $\phi = P(\text{Scores } 100\%)$ while $\theta = P(\text{Scores over } 90)$ 1.

Since $p_{00} + p_{01} + p_{02} = 1$, $p_{01} = 1 - (1-\theta) - \phi = \theta - \phi$ 1.

$p_{11} = 0$ since, from state 1, a candidate can either score under 90%, so back to 0, or gains the Certificate (either a 2nd consecutive 90-99 or a perfect 100%). 1.

/4 $p_{22} = 1$ as once the Certificate has been awarded it stays awarded. 1.

(ii) Let $N = N^o$ steps to absorption in 2. Start in state 0.

$\tau_0 = \mathbb{E}[N \mid X_1 = 0](1-\theta) + \mathbb{E}[N \mid X_1 = 1](\theta - \phi) + 1$ 1.

$= (1-\theta)\tau_0 + (\theta - \phi)\tau_1 + 1$ 1.

/4 $\tau_1 = \mathbb{E}[N \mid X_1 = 0](1-\theta) + 1 = (1-\theta)\tau_0 + 1$

(iii) $\therefore [1 - (1-\theta)]\tau_0 = (\theta - \phi)[(1-\theta)\tau_0 + 1] + 1$ 1.

$\therefore [\theta - (\theta - \phi)(1-\theta)]\tau_0 = \theta - \phi + 1$

/2 $\therefore \tau_0 = \dfrac{1 + \theta - \phi}{\theta^2 + \phi(1-\theta)}$ 1.

(iv) $\phi = 0.05$, $1 - \theta = 0.75$ so $\theta = 0.25$ 1.

$\therefore \tau_0 = \dfrac{1 + 0.2}{0.0625 + 0.05 \times 0.75} = \dfrac{1.2}{0.1} = 12$ 1.

/5 $\tau_1 = 0.75 \times 12 + 1 = 10$ 1.

so $\tau_0 - \tau_1 = 2$. 1.

By the Markov property the required difference is 2 attempts 1.

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2. (i)(a) $p_i = P(A \text{ wins})\,P(A \text{ ruined} \mid X_A = i+1) + P(A \text{ loses})\,P(A \text{ ruined} \mid X_A = i-1)$ 1. 2.

$= \underline{\gamma p_{i+1} + (1-\gamma) p_{i-1}}$

$p_0 = P(A \text{ ruined} \mid X_A = 0) = \underline{1}$, since if $X_A = 0$ then A is already ruined. 1. 1.

$p_T = P(A \text{ ruined} \mid X_A = T) = \underline{0}$, since if $X_A = T$ then B is already ruined. 1. 1.

/6

(b) Substituting into RS yields

$\gamma\left(A + B\left(\frac{1-\gamma}{\gamma}\right)^{i+1}\right) + (1-\gamma)\left(A + B\left(\frac{1-\gamma}{\gamma}\right)^{i-1}\right)$ 1.

$= \gamma A + (1-\gamma)A + B\frac{(1-\gamma)^{i+1}}{\gamma^i} + B\frac{(1-\gamma)^i}{\gamma^{i-1}}$

$= A + B\left(\frac{1-\gamma}{\gamma}\right)^i \{(1-\gamma) + \gamma\}$ 1

$= \underline{A + B\Phi^i = p_i}$, as required 1.

/3

(c) Now $p_0 = 1 \Rightarrow A + B = 1 \Rightarrow A = 1 - B$ 1.

while $p_T = 0 \Rightarrow A + B\Phi^T = 0 \Rightarrow B(\Phi^T - 1) = -1$ 1.

$\Rightarrow B = -1/(\Phi^T - 1)$ 1.

$\therefore p_i = 1 + \frac{1}{\Phi^T - 1} - \frac{\Phi^i}{\Phi^T - 1} = \frac{\Phi^T - \Phi^i}{\Phi^T - 1}$ 1.

Hence, at start of game, P(Amy ruined) is

$p_a = \underline{\frac{\Phi^T - \Phi^a}{\Phi^T - 1}}$ 1.

/5

(ii) Now $b \to \infty$ so $T \to \infty$. 1.

For $0 < \gamma < 0.5$, $1-\gamma > \gamma \Rightarrow \Phi > 1$ 1.

$\therefore p_a = \frac{1 - \Phi^{a-T}}{1 - \Phi^{-T}} \to \underline{1}$, as $T \to \infty$. 1. 1.

/6

For $0.5 < \gamma < 1$, $1-\gamma < \gamma \Rightarrow 0 < \Phi < 1$ 1.

$\therefore p_a \to \frac{0 - \Phi^a}{0 - 1} = \underline{\Phi^a}$, as $T \to \infty$. 1.

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3(i) $p_n = P(N(t)=n) = \underline{e^{-\lambda t}(\lambda t)^n/n!}$, $n=0,1,2,\ldots$ 1.

$P(T_1 \le t) = 1 - P(N(t)=0) = \underline{1-e^{-\lambda t}}$, $t>0$ 1.

/3 which is the cdf of $\underline{\text{Expon}(\lambda)}$. 1.

(ii) $P(T_1 < s \mid N(t)=1) = \dfrac{P(N(s)=1, N(s,t)=0)}{P(N(t)=1)}$ 1, 1

$= \dfrac{\lambda s e^{-\lambda s} \times e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \underline{\underline{\dfrac{s}{t}}}$ 1, 1

Pdf is $f(s) = \dfrac{d}{ds}\left(\dfrac{s}{t}\right) = \underline{\underline{\dfrac{1}{t}}}$, $0<s<t$

so $T_1 \mid N(t)=1 \sim \underline{\underline{U[0,t]}}$ 1

/4 $E[N(t)] = \underline{\lambda t}$ and $E[T_1 \mid N(t)=1] = \underline{\underline{\dfrac{t}{2}}}$ 1, 1

Result given → ($S_1, \ldots, S_n$ are jointly distributed as n independent Uniform rvs, $U[0,t]$.)

(iii) Expected N^o in $[0,T] = \lambda T$. Each has an 1.

expected wait $= T - E[S_i] = T - \dfrac{T}{2} = \dfrac{T}{2}$ by (iii). 1.

/3 $\therefore E[\text{Total wait}] = \lambda T \times \dfrac{T}{2} = \underline{\underline{\dfrac{\lambda T^2}{2}}}$ 1.

(iv) (a) Applying (iii) to the intervals $[0,u]$, (u,v), $(v,T]$

$E[\text{Total wait}] = \dfrac{\lambda u^2}{2} + \dfrac{\lambda (v-u)^2}{2} + \dfrac{\lambda (T-v)^2}{2}$ 1.

/2 $= \underline{\underline{\dfrac{\lambda}{2}\{u^2 + (v-u)^2 + (T-v)^2\}}}$ 1.

(b) Minimise $h(u,v) = u^2 + (v-u)^2 + (T-v)^2$

$\dfrac{\partial h}{\partial u} = 2u - 2(v-u) = 0 \Rightarrow u = \dfrac{v}{2}$ 1, 1

/5 $\dfrac{\partial h}{\partial v} = 2(v-u) - 2(T-v) \ (=0) \Rightarrow v - \frac{1}{2}v + v = T$ 1.

$\Rightarrow v = \underline{\dfrac{2}{3}T}$ and $u = \underline{\dfrac{1}{3}T}$ for any λ 1

Since $\dfrac{\partial^2 h}{\partial u^2} = 4 = \dfrac{\partial^2 h}{\partial v^2}$ and $\dfrac{\partial^2 h}{\partial u \partial v} = -2$

20 Det $\left[\dfrac{\partial^2 h}{\partial \cdot \partial \cdot}\right] = 16 - 4 = 12 > 0 \Rightarrow$ minimum 1.

4(i) Consider the number, N_S, in the system as customer departs. These all arrived during customer's time in system, T_S. 1.

$\therefore E[N_S | T_S = t] = \lambda t$ so $E[N_S] = \lambda E[T_S]$ 1.

and hence $E[T_S] = \frac{1}{\lambda} E[N_S]$. 1.

/3

[1 mark only for citing Little's formulae.]

(ii) (a) $\frac{1}{\mu} = 3$, $\lambda = \frac{1}{4}$ per hour, $\rho = \frac{3}{4}$ and $\sigma^2 = 0$. 1.

$\therefore E[N_S] = \dfrac{\frac{3}{4}(2 - \frac{3}{4}) + 0}{2(1 - \frac{3}{4})} = 15/8$ 1.

/3

$\therefore E[T_S] = 15/8 \div \frac{1}{4} = \underline{15/2}$ or $\underline{7\frac{1}{2}}$ hours 1.

(b) $X = X_1 + X_2 \Rightarrow \frac{1}{\mu} = E[X] = \frac{3}{2} + \frac{3}{2} = 3 \Rightarrow \rho = \frac{3}{4}$ 1,1

$\sigma^2 = \text{Var}\, X = \text{Var}\, X_1 + \text{Var}\, X_2 = 2 \times (\frac{3}{2})^2 = 9/2$ 1

$\therefore E[N_S] = \dfrac{\frac{3}{4}(2 - \frac{3}{4}) + (\frac{1}{4})^2 \frac{9}{2}}{2(1 - \frac{3}{4})} = \dfrac{\frac{15}{16} + \frac{9}{32}}{\frac{1}{2}}$ 1.

/5

$= 39/16 \Rightarrow E[T_S] = \underline{39/4} = \underline{9\frac{3}{4}}$ hours 1.

(c) $X = \max(X_1, X_2)$ where $X_1, X_2 \sim$ independent Expon$(\frac{2}{3})$ 1.

$\therefore F_X(x) = P(X_1 \le x) \times P(X_2 \le x) = (1 - e^{-\frac{2}{3}x})^2$

$= 1 - 2e^{-\frac{2}{3}x} + e^{-\frac{4}{3}x}$ 1

$\therefore f_X(x) = \frac{4}{3} e^{-\frac{2}{3}x} - \frac{4}{3} e^{-\frac{4}{3}x}$ 1.

Using $\int_0^\infty x^r \theta e^{-\theta x} dx = \int_0^\infty \dfrac{y^r e^{-y}}{\theta^r} dy = r!/\theta^r$ 1.

$\therefore E[X^r] = 2\int_0^\infty x^r \tfrac{2}{3} e^{-\frac{2}{3}x} dx - \int_0^\infty x^r \tfrac{4}{3} e^{-\frac{4}{3}x} dx$

$= 2\, r!/(\tfrac{2}{3})^r - r!/(\tfrac{4}{3})^r$

$\therefore \frac{1}{\mu} = E[X] = 2 \times \frac{3}{2} - \frac{3}{4} = 9/4$ so $\rho = 9/16$ 1.

$E[X^2] = 2 \times 2 \times (\frac{3}{2})^2 - 2 \times (\frac{3}{4})^2 = 63/8$ 1

$\therefore \sigma^2 = 63/8 - (9/4)^2 = 45/16$ 1.

/9

$\therefore E[N_S] = \dfrac{\frac{9}{16}(2 - \frac{9}{16}) + (\frac{1}{4})^2 \frac{45}{16}}{2(1 - \frac{9}{16})} = \dfrac{9 \times 23 + 45}{16 \times 14}$

$= 9/8$ 1.

$\therefore E[T_S] = \underline{9/2} = \underline{4\frac{1}{2}}$ hours. 1.

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5. (i) $Y_{t+k} = Y_{t+k-1} + A_{t+k} - \beta A_{t+k-1}$, for $k \geq 1$ 1.

$\hat{Y}_t(k) = E[Y_{t+k} \mid Y_t, Y_{t-1}, \ldots]$

$\therefore$ Since $E[A_{t+k} \mid Y_t, Y_{t-1}, \ldots] = 0$ for $k \geq 1$

$\hat{Y}_t(k) = \hat{Y}_t(k-1) = \hat{Y}_t(1)$, for $k \geq 2$ (constant) 1.

and $\hat{Y}_t(1) = Y_t - \beta a_t$ where $a_t = E[A_t \mid Y_t, Y_{t-1}, \ldots]$ 1.

4 $a_t = Y_t - \hat{Y}_{t-1}(1)$ the one-step forecast error at $t-1$. 1.

(ii) $(1-B)Y_t = (1-\beta B)A_t$ with $BY_t = Y_{t-1}$ 1

$\therefore Y_t = \frac{(1-\beta B)}{1-B} A_t = (1+B+B^2+\ldots)(1-\beta B)A_t$ 1

$= (1 + (1-\beta)B + (1-\beta)B^2 + \ldots)A_t$ 1

Hence $\psi_0 = 1$, $\psi_i = 1-\beta$ for $i \geq 1$. 1.

Now $V(k) = \mathrm{Var}(Y_{t+k} - \hat{Y}_t(k) \mid Y_t, Y_{t-1}, \ldots)$

$= \sigma^2 \{\psi_0^2 + \psi_1^2 + \ldots + \psi_{k-1}^2\}$. 1.

7 $\therefore V(1) = \sigma^2$, $V(k) = \sigma^2\{1 + (k-1)(1-\beta)^2\}$, 1,1

$\| k \geq 1$.

(iii) a. Now $V(1) = 16$ so $\sigma^2 = 16$ 1

and $V(2) = 25$ so $1 + (1-\beta)^2 = \frac{25}{16}$ 1

3 $\therefore \beta = \frac{1}{4}$ 1

b. Forecast error $= (Y_{t+1} + Y_{t+2}) - (\hat{Y}_t(1) + \hat{Y}_t(2)) = e_t(1) + e_t(2)$

$= \{Y_{t+1} - \hat{Y}_t(1)\} + \{\hat{Y}_{t+2} - \hat{Y}_t(2)\}$ 1.

$= A_{t+1} + A_{t+2} + \psi_1 A_{t+1} = A_{t+2} + (2-\beta)A_{t+1}$ 1

$= A_{t+2} + \frac{7}{4} A_{t+1}$ 1.

$\therefore$ Forecast error variance $= \sigma^2 + \frac{49}{16}\sigma^2 = 65$ 1.

since A_{t+1}, A_{t+2} uncorrelated.

$\therefore$ 90% prediction interval is

$2 \times 812 \pm 1.645 \times \sqrt{65}$ (± 13.26) 1.

6 ie $(1610.74, 1637.26)$ assuming forecast error approximately Normal. 1.

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6(i) $\rho_k = \dfrac{Cov(X_t, X_{t+k})}{Var(X_t)}$ for $k = 0, 1, 2, \ldots$ 1.

(ii) r_k is the sample autocorrelation, an estimator of ρ_k. The formula assumes $\rho_k = 0$ for $k = 1, 2, 3, \ldots$ ie valid for white noise. 1.

Comparing $|r_k|$ with $\pm 2\,SE(r_k)$ for $k = 1, 2, 3, \ldots$ provides a test of $\rho_k \neq 0$. 1.

Since, for MA(q), $\rho_k = 0$ for $k \geq q \neq 1$ we can rule out successively white noise, MA(1), MA(2) etc 1.

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(iii) ACF $SE(r_1) \simeq \sqrt{\frac{1}{100}} = \underline{0.1}$, $SE(r_2) \simeq \sqrt{(1 + 2 \times 0.654^2)/100} = \underline{0.136}$

$SE(r_3) \simeq \sqrt{(1 + 2 \times 0.654^2 + 2 \times 0.308^2)/100} = \underline{0.143}$

PACF $SE(\hat{\phi}_{kk}) \simeq \sqrt{1/100} = \underline{0.1}$ 4×0.5

$r_1 = 0.654 > 2 \times 0.1$ ruling out white noise. 1.

$r_2 = 0.308 > 2 \times 0.136$ so not MA(1). 1.

For all $k = 3, 4, \ldots, 16$, $|r_k| < 2 \times 0.143 = 0.286$ so MA(2) is a plausible candidate. 1.

6. $|\phi_{kk}| > 2 \times 0.1$ for $k = 1, 2$ and 5 which rules out AR(1). 1.

(and AR(p) for $p = 2, 3, 4$)

(iv) A MA(2) or ARIMA(0, 0, 2) 1.

$X_t = 11.1651 + A_t + 0.8084 A_{t-1} + 0.4759 A_{t-2}$ 1.

B ARIMA(3, 1, 0). With $Y_t = X_t - X_{t-1}$ 1, 1

5 $Y_t = 0.0133 - 0.0525 Y_{t-1} - 0.2091 Y_{t-2}$

$- 0.2844 Y_{t-3} + A_t$ 1.

(v) A All parameters are significantly different from 0 and none of the Box-Ljung stats are. So MA(2) seems to be adequate. 1.

1.

B The plot does not justify differencing and the fitted constant (though NS) would imply a linear trend. Box-Ljung stats. are significant at lags 12 and 48 so not a suitable model. 1.

4

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7.(i) $\mu = \mathbb{E}[Y_t]$, $\gamma_0 = \mathrm{Var}[Y_t]$ and $\rho_k = \mathrm{Corr}(Y_t, Y_{t-k})$ are all finite and do not depend on t. 1. 1. 1/3

(ii) $\mathbb{E}[Z_t] = \mathbb{E}[Y_t] + \omega\mathbb{E}[Y_{t-1}] = (1+\omega)\mu$ ---(1) 1.

$\mathrm{Var}[Z_t] = \mathrm{Var}[Y_t] + \omega^2\mathrm{Var}[Y_{t-1}] + 2\omega\mathrm{Cov}[Y_t, Y_{t-1}]$ 1.
$= \gamma_0 + \omega^2\gamma_0 + 2\omega\gamma_1$ with $\gamma_1 = \mathrm{Cov}[Y_t, Y_{t-1}]$
$= [1 + \omega^2 + 2\omega\rho_1]\gamma_0$, since $\rho_1 = \gamma_1/\gamma_0$ 1.
---(2)

$\mathrm{Cov}[Z_t, Z_{t-k}] = \mathrm{Cov}[Y_t + \omega Y_{t-1}, Y_{t-k} + \omega Y_{t-k-1}]$ 1.
$= \mathrm{Cov}[Y_t, Y_{t-k}] + \omega^2\mathrm{Cov}[Y_{t-1}, Y_{t-k-1}]$
$+ \omega\mathrm{Cov}[Y_t, Y_{t-k-1}] + \omega\mathrm{Cov}[Y_{t-1}, Y_{t-k}]$ 1.
$= (1+\omega^2)\gamma_k + \omega(\gamma_{k+1} + \gamma_{k-1})$ 1.

$\therefore \rho_Z(k) = \dfrac{\mathrm{Cov}[Z_t, Z_{t-k}]}{\mathrm{Var}[Z_t]} = \dfrac{(1+\omega^2)\gamma_k + \omega(\gamma_{k+1}+\gamma_{k-1})}{\gamma_0[1+\omega^2+2\omega\rho_1]}$ 1.

$= \dfrac{(1+\omega^2)\rho_k + \omega(\rho_{k+1}+\rho_{k-1})}{1+\omega^2+2\omega\rho_1}$ ---(3) 1.

since $\rho_k = \gamma_k/\gamma_0$.

Since Y_t is 2nd order stationary, (1), (2) and (3) are all finite and are not dependent on t. Hence Z_t is also 2nd order stationary. 1. 1/9

(iii) Taking $\omega = -1$ in part (ii) we get $Z_t = Y_t - Y_{t-1}$ 1.
stationary. Applying (ii) to Z_t we get $V_t = Z_t - Z_{t-1}$ 1.
stationary and $V_t = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2})$. 1. 1/3

(iv) For white noise we require $\rho_Z(k) = 0$ for all $k \geq 1$. 1.
$\therefore (1+\omega^2)\rho_k + \omega(\rho_{k+1} + \rho_{k-1}) = 0$, from (3)
$\therefore (1+\omega^2)\alpha^k + \omega(\alpha^{k+1} + \alpha^{k-1}) = 0$ 1.
Assuming $\alpha \neq 0$, $(1+\omega^2)\alpha + \omega(\alpha^2+1) = 0$ 1.
$\therefore \alpha\omega^2 + (1+\alpha^2)\omega + \alpha = 0$
$\therefore (\alpha\omega + 1)(\omega + \alpha) = 0$ 1.
Hence $\omega = -\alpha$ or $-1/\alpha$ 1. 1/5

8. (i) $f(\omega) = \frac{1}{2\pi}\left\{\sum_{k=-\infty}^{-1} e^{-i\omega k}\gamma_k + \gamma_0 + \sum_{k=1}^{\infty} e^{-i\omega k}\gamma_k\right\}$ 1.

$\therefore g(\omega) = \frac{2\pi f(\omega)}{\gamma_0} = 1 + \sum_{\ell=1}^{\infty} e^{i\omega\ell}\frac{\gamma_{-\ell}}{\gamma_0} + \sum_{k=1}^{\infty} e^{-i\omega k}\frac{\gamma_k}{\gamma_0}$ 1,1

$= 1 + \sum_{k=1}^{\infty}(e^{i\omega k} + e^{-i\omega k})\rho_k$, since $\gamma_{-k} = \gamma_k$ and $\gamma_k/\gamma_0 = \rho_k$ 1,1

/5

(ii) $g(-\omega) = 1 + \sum_{k=1}^{\infty}(e^{-i\omega k} + e^{i\omega k})\rho_k = g(\omega)$ for $0 < \omega < \pi$. 1,1

/2

(iii) $\gamma_k = Cov(X_t, X_{t-k}) = Cov(\alpha X_{t-1} + A_t, X_{t-k})$ 1.

$= \alpha\gamma_{k-1} + 0$ since A_t and X_{t-k} are uncorrelated for $k \geq 1$. 1.

$= \alpha^k\gamma_0$ 1.

/4 Hence $\rho_k = \alpha^k$, $k = 0, 1, 2, \ldots$ 1.

$\therefore g(\omega) = 1 + \sum_{k=1}^{\infty}(e^{-i\omega}\alpha)^k + \sum_{k=1}^{\infty}(e^{i\omega}\alpha)^k$ 1.

$= 1 + \frac{\alpha e^{-i\omega}}{1-\alpha e^{-i\omega}} + \frac{\alpha e^{i\omega}}{1-\alpha e^{i\omega}}$ 1.

$= 1 + \frac{\alpha e^{-i\omega}(1-\alpha e^{i\omega}) + \alpha e^{i\omega}(1-\alpha e^{-i\omega})}{1 - \alpha(e^{-i\omega} + e^{i\omega}) + \alpha^2}$ 1.

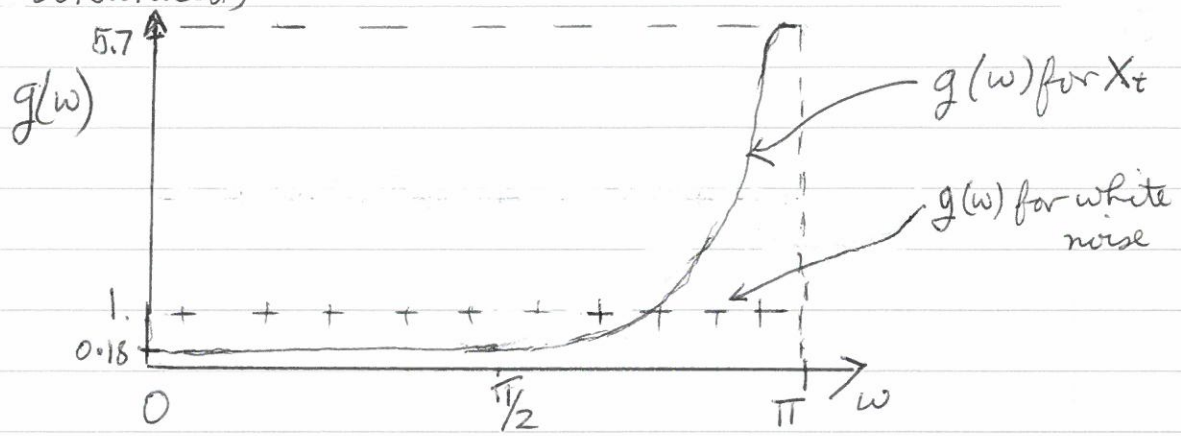
$= \frac{1 - 2\alpha\cos\omega + \alpha^2 + 2\alpha\cos\omega - 2\alpha^2}{1 - 2\alpha\cos\omega + \alpha^2}$

/4 $= \frac{1-\alpha^2}{1 - 2\alpha\cos\omega + \alpha^2}$ 1.

(iv) For $\alpha = -0.7$, $g(\omega) = \frac{0.51}{1.49 + 1.4\cos\omega}$

Since $\cos\omega$ is decreasing over $[0, \pi]$, $g(\omega)$ will be an increasing function from $g(0) = 0.18$ to $g(\pi) = 5.7$.
(alternatively, $g'(\omega) = 0.51(1.49 + 1.4\cos\omega)^{-2} \times 1.4\sin\omega$
> 0 so $g(\cdot)$ increasing (and $g'(\pi) = 0$!)
(Sketch on next p.)

8. (iv continued)



3 (*)

The equivalent function for white noise is constant (ie 1) 1.
Since a higher proportion of the variance is at high frequencies compared to white noise, realisations of X_t will be more jagged in appearance. 1.

(*) [Credit: .1 mark for increasing function on $[0, \pi]$
1 " " justification of increasing
1 " for range of $g(\cdot)$]

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