EXAMINATIONS OF THE HONG KONG STATISTICAL SOCIETY

HIGHER CERTIFICATE IN STATISTICS, 2006

Paper I : Statistical Theory

Time Allowed: Three Hours

Candidates should answer **FIVE** *questions.*

All questions carry equal marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use calculators in accordance with the regulations published in the Society's "Guide to Examinations" (document Ex1).

The notation log *denotes logarithm to base* ***e***.
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^nC_r$.

This examination paper consists of 6 printed pages **each printed on one side only**.
This front cover is page 1.
Question 1 starts on page 2.

There are 8 questions altogether in the paper.

1. Credit card holders with a certain bank are each assigned a 4-digit PIN (personal identity number). How many different possible PINs are there under each of the following conditions?

(i) The first digit may not be zero.
(3)

(ii) The first digit may not be zero, and the four digits may not all be the same.
(3)

(iii) No zeros are allowed in any position and all four digits must be different.
(4)

(iv) Zeros are allowed in all positions but sequences of four consecutive digits up (e.g. 2 3 4 5) or down (e.g. 3 2 1 0) are not allowed.
(4)

(v) Zeros are allowed in all positions but no digit may occur more than twice.
(6)

2. In a hi-tech company, the members of three research groups (A, B and C) are individually invited to enter a prize competition for the best solution to a technical problem. Group A has 2 staff, B has 3 and C has 5. It is assumed that all staff decide independently whether or not to enter. Members of groups A, B and C enter with respective probabilities 1/2, 1/4 and 1/5.

(i) For each group separately, find the probability of (a) no entries, (b) one entry.
(8)

(ii) Given that there is just one entry in total, show that the probability that it comes from a member of group A is 8/17.
(6)

(iii) Explain (but without doing the calculations) the steps that are needed to calculate the probability that there are exactly two entries in total.
(6)

3. The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} kx^2(1-x)^2, & 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(i) Find k and sketch the graph of $f(x)$.

(7)

(ii) Find $E(X)$ and $\text{Var}(X)$, and show that $P\left(X \le \frac{1}{3}\right) = \frac{17}{81}$.

(8)

(iii) A random sample of size 5 is taken from this distribution. Find, correct to 4 decimal places, the probability that all 5 observations exceed 1/3.

(3)

(iv) Find, correct to 4 decimal places, the variance of the mean of a random sample of size 5.

(2)

4. My cycle journey to work is 3 km, and my cycling time (in minutes) if there are no delays is distributed N(15, 1), i.e. Normally with mean 15 and variance 1.

(i) Find the probability that, if there are no delays, I get to work in at most 17 minutes.

(2)

(ii) On my route there are three sets of traffic lights. Each time I meet a red traffic light, I am delayed by a random time that is distributed N(0.7, 0.09). These lights operate independently. Find the probability of my getting to work in at most 17 minutes

(a) if just one light is set at red when I reach it,

(b) if just two lights are set at red when I reach them,

(c) if all three lights are set at red when I reach them.

(9)

(iii) Suppose that, for each set of lights, the chance of delay is 0.5. Deduce that the mean value of T, my total journey time, is 16.05 minutes.

(4)

(iv) Given that $\text{Var}(T) = 1.5025$, use a suitable approximation to calculate the probability that, over 10 journeys, my average journey time to work is at most 17 minutes.

(5)

5. The random variable X follows a Poisson distribution with probability mass function

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots\ .$$

(i) Show that $E(X) = \text{Var}(X) = \lambda$.

(5)

(ii) Given a random sample of values $x_1, x_2, \ldots, x_n$ from this distribution, obtain the maximum likelihood estimator (MLE) of λ, $\hat{\lambda}$ say.

(4)

(iii) Write down $\text{Var}(\hat{\lambda})$ as a function of λ, and hence find the MLE of $\text{Var}(\hat{\lambda})$. Show that a large-sample approximate 95% confidence interval (CI) for λ is given by

$$\hat{\lambda} - 1.96\sqrt{\frac{\hat{\lambda}}{n}} \le \lambda \le \hat{\lambda} + 1.96\sqrt{\frac{\hat{\lambda}}{n}}\ .$$

(4)

(iv) Assume that the numbers of books, $x_1, x_2, x_3, \ldots$, that go missing each month from the local library follow a Poisson distribution with unknown mean λ. The monthly numbers of missing books in 2005 were

3 7 2 5 8 2 4 5 4 4 1 3.

Use these data to calculate $\hat{\lambda}$ and an approximate 95% CI for λ. Also compute the sample variance of the data; discuss briefly whether this computation supports, or throws doubt on, the Poisson model suggested (no formal test is required).

(7)

6. The random variable X has the distribution with probability density function

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \qquad -\infty < x < \infty.$$

Sketch a graph of this density function.

(4)

Write down $E(X)$ and show that $\text{Var}(X) = \frac{2}{\lambda^2}$. Find also the semi-interquartile range of X.

(9)

A random sample $x_1, x_2, \ldots, x_n$ is taken from this distribution. Show that the maximum likelihood estimate of λ is given by

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} |x_i|}.$$

(7)

7. The table below shows the joint distribution of two random variables, X and Y.

		Values of Y			
		1	2	3	4
Values of X	1	$6c$	$3c$	$2c$	$4c$
	2	$4c$	$2c$	$4c$	0
	3	$2c$	c	0	$2c$

(i) Find c.

(2)

(ii) Calculate the marginal distributions of X and Y.

(5)

(iii) Calculate $E(X)$ and $\text{Var}(X)$, and show that the covariance $\text{Cov}(X, Y) = 0$.

(6)

(iv) State, with a reason, whether or not X and Y are independent.

(2)

(v) The random variables U and V are defined by

$U = 1$ if $X = 1$ or 3, $U = 0$ if $X = 2$,

$V = 1$ if $Y = 1$ or 3, $V = 0$ if $Y = 2$ or 4.

Tabulate the joint distribution of U and V and state with a reason whether or not U and V are independent.

(5)

8. (i) Write down the model for, and standard assumptions of, simple linear regression analysis. State a condition under which the method of least squares is equivalent to the method of maximum likelihood for estimating the regression coefficients.

(4)

(ii) (a) Suppose now that the intercept parameter in the regression model is known to be zero, so that the model becomes

$$y_i = \beta x_i + e_i,$$

where the usual assumptions apply to e_i. Show that the least squares estimator of β is $\dfrac{\sum x_i y_i}{\sum x_i^2}$.

(6)

(b) Over a period of one month, a survey was made on each of ten main roads in a large city. Each road was observed for a one-hour period randomly chosen during the working day. For each road, the mean traffic flow, x_i (in vehicles per minute), and the number of speed limit violations, y_i, $i = 1, 2, \ldots, 10$, were recorded. Plot the data shown below on a suitable graph and comment on the suitability of the above model. Fit the model to the data and hence estimate the expected number of violations on a road with an average traffic flow of 20 vehicles per minute. *Without* any further calculation, comment on the suggestion that an intercept should be included in the model.

Flow, x	5	5	5	10	10	15	25	25	30	50
Violations, y	2	1	1	4	2	5	8	2	5	10

(10)