HONG KONG STATISTICAL SOCIETY

2016 EXAMINATIONS – SOLUTIONS

GRADUATE DIPLOMA – MODULE 1

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The solutions are intended as learning aids and should not be seen as "model answers".

Users of the solutions should always be aware that in many cases there are valid alternative methods. Also, in the many cases where discussion is called for, there may be other valid points that could be made.

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Question 1

(i)	$\begin{array}{c\|cccc} u & 1 & 2 & \ldots & k \\ \hline p(u) & \frac{1}{k} & \frac{1}{k} & & \frac{1}{k} \end{array}$ [*distribution might be implicit*]	**1**
	So $E(U) = \frac{1}{k}(1 + \ldots + k) = \frac{1}{k}.\tfrac{1}{2}k(k+1)$ [*1 mark*] $= \tfrac{1}{2}(k+1)$ [*1 mark*]	**1, 1**
	$E(U^2) = \frac{1}{k}(1^2 + \cdots + k^2) = \frac{1}{k}.\frac{k(k+1)(2k+1)}{6} = \frac{1}{6}(k+1)(2k+1)$	**1**
	$\text{Var}(U) = \frac{1}{6}(k+1)(2k+1) - \frac{1}{4}(k+1)^2 = \frac{1}{12}(k^2-1)$	**1, 1**
(ii)	(a) The range space of S is $\{2, 3, \ldots\}$. For any s in this range,	**1**
	$P(S = s) = \sum_{x=1}^{s-1} P(X = x \text{ and } Y = s - x)$	**1**
	$= \sum_{x=1}^{s-1} P(X = x).P(Y = s - x)$ [independe nce]	**1, 1**
	$= \sum_{x=1}^{s-1} (1-\theta)^{x-1}\theta(1-\theta)^{s-x-1}\theta$	**1**
	$= (s-1)(1-\theta)^{s-2}\theta^2$	**1**
	(b) Suppose that $S = s$, for some s greater than or equal to 2. Then X must take values in the range $1, 2, \ldots, s - 1$. For any x in this range,	**1**
	$P(X = x \mid S = s) = \frac{P(X = x \text{ and } Y = s - x)}{P(S = s)}$	**1**
	$= \frac{(1-\theta)^{x-1}\theta(1-\theta)^{s-x-1}\theta}{(s-1)\theta^2(1-\theta)^{s-2}}$	**1**
	$= \frac{1}{s-1}$	**1**
	So, conditional on $S = s$, X has the distribution of (i) with $k = s - 1$.	**1**
	$E(X \mid S = s) = \frac{s}{2}$, $\text{Var}(S) = \frac{1}{12}((s-1)^2 - 1) = \frac{s(s-2)}{12}$	**1, 1, 1**

Question 2

(i)	$E(U)=\int_0^\infty u\,\frac{\theta^n u^{n-1}e^{-\theta u}}{(n-1)!}\,du$	
	$=\frac{\theta^n}{(n-1)!}\int_0^\infty u^n e^{-\theta u}\,du$	
	$=\frac{\theta^n}{(n-1)!}\frac{1}{\theta^{n+1}}\int_0^\infty t^n e^{-t}dt$ [where $t=\theta u$]	**1**
	$=\frac{\theta^n}{(n-1)!}\frac{n!}{\theta^{n+1}}$	**1**
	$=\frac{n}{\theta}$	**1**
	Similarly, $E(U^2)=\frac{n(n+1)}{\theta^2}$. [*Give marks also if full working shown.*]	**2**
	So, $\text{Var}(U)=\frac{n(n+1)}{\theta^2}-\frac{n^2}{\theta^2}=\frac{n}{\theta^2}$.	**1**
(ii)	(a) $f(x)=\int_x^\infty 4xe^{-(x+y)}\,dy=4xe^{-x}\int_x^\infty e^{-y}\,dy=4xe^{-2x},\quad x>0$	**1, 1, 1**
	Using part (i), with n = 2 and θ = 2, $E(X)$ = 1, Var(X) = ½.	**1, 1**
	(b) $f(y\mid x)=\frac{4xe^{-(x+y)}}{4xe^{-2x}}=e^{-(y-x)},\quad y>x$	**1, 1**
	$\therefore$ $E(Y\mid X=x)=\int_x^\infty ye^{-(y-x)}\,dy$	**1**
	$=\int_0^\infty (t+x)e^{-t}\,dt$ $[t=y-x]$	**1**
	$=\int_0^\infty te^{-t}\,dt+x\int_0^\infty e^{-t}\,dt$	
	$=1!+x(0!)$	**1**
	$=1+x$	**1**
	(c) Using the Law of Iterated Expectation,	
	$E(Y)=E\{E(Y\mid X)\}$	**1**
	$=E(1+X)$	
	$=1+E(X)$	**1**
	$=2$	**1**

Question 3

(i)	$F(z)=\int_0^z \alpha\theta t^{\theta-1}\exp\left(-\alpha t^{\theta}\right)dt$	**1**
	$=\int_0^{\alpha z^{\theta}} e^{-u}du \qquad [u=\alpha t^{\theta}]$	**1**
	$=1-\exp(-\alpha z^{\theta})$	**1**
	so $\quad h(z)=\dfrac{f(z)}{1-F(z)}=\dfrac{\alpha\theta z^{\theta-1}\exp(-\alpha z^{\theta})}{\exp(-\alpha z^{\theta})}=\alpha\theta z^{\theta-1}$	**1**
	(a) $h(z)$ constant for $z \geq 0$ if $\theta = 1$, (b) $h(z)$ decreases with $z \geq 0$ if (0<) $\theta < 1$ [*Don't need to state $\theta > 0$ to get the second of these marks.*]	**1, 1**
(ii)	$G(y)$ = P(either component fails in time y)	
	$= P(X_1 \leq y \text{ or } X_2 \leq y)$	**1**
	$= P(X_1 \leq y) + P(X_2 \leq y) - P(X_1 \leq y \text{ and } X_2 \leq y)$	
	$= P(X_1 \leq y) + P(X_2 \leq y) - P(X_1 \leq y).P(X_2 \leq y)$ [independence]	**1**
	$= F_1(y) + F_2(y) - F_1(y).F_2(y)$	**1**
	[*Give full marks to candidate who realises the survivor function of Y must be the product of the survivor functions of X_1 and X_2.*]	
	$g(y) = G'(y) = f_1(y) + f_2(y) - f_1(y).F_2(y) - F_1(y).f_2(y)$	**1**
	$\therefore \quad h(y)=\dfrac{g(y)}{1-G(y)}=\dfrac{f_1(y)[1-F_2(y)]+f_2(y)[1-F_1(y)]}{[1-F_1(y)][1-F_2(y)]}=h_1(y)+h_2(y)$	**1**
	In the Weibull case,	
	$h(y)=h_1(y)+h_2(y)=\alpha_1\theta y^{\theta-1}+\alpha_2\theta y^{\theta-1}=(\alpha_1+\alpha_2)\theta y^{\theta-1}$	**1, 1**
	This is the hazard function of another Weibull distribution.	**1**
(iii)	$G(y)$ = P(C_1 and C_2 fail in time y) = $F_1(y).F_2(y)$ [*independence*]	**1, 1**
	When the two components are identical,	
	$G(y) = [F(y)]^2$	**1**
	and $\quad g(y) = 2.F(y).f(y)$	**1**
	So $\quad h(y)=\dfrac{2F(y)f(y)}{1-[F(y)]^2}=2.\dfrac{F(y)}{1+F(y)}.\dfrac{f(y)}{1-F(y)}\leq\dfrac{f(y)}{1-F(y)}$	**1**
	since $\quad \dfrac{F(y)}{1+F(y)}\leq\dfrac{1}{2}$ because $0 \leq F(y) \leq 1$.	**1**

Question 4

(a)	(i) $\text{Cov}(X_1, X_2) = \rho\,\sqrt{[\text{Var}(X_1).\text{Var}(X_2)]} = -20$	**1**
	So $E(\underline{X}) = (50 \quad 45)^T$ and $\text{Cov}(\underline{X}) = \begin{pmatrix} 64 & -20 \\ -20 & 100 \end{pmatrix}$	**1, 1**
	(ii) The random vector Y has a bivariate normal distribution, with	**1**
	$E(\underline{Y}) = \begin{pmatrix} 0.555 & 0 \\ 0 & 0.447 \end{pmatrix}\begin{pmatrix} 50 \\ 45 \end{pmatrix} + \begin{pmatrix} -17.76 \\ 0 \end{pmatrix} = \begin{pmatrix} 9.99 \\ 20.12 \end{pmatrix}$	**1, 1**
	$\text{Cov}(\underline{Y}) = \begin{pmatrix} 0.555 & 0 \\ 0 & 0.447 \end{pmatrix}\begin{pmatrix} 64 & -20 \\ -20 & 100 \end{pmatrix}\begin{pmatrix} 0.555 & 0 \\ 0 & 0.447 \end{pmatrix}$	**1**
	$= \begin{pmatrix} 19.71 & -4.96 \\ -4.96 & 19.98 \end{pmatrix}$	**1**
	[*Candidates who use 20.115 instead of 20.12 in these calculations should be awarded full marks.*]	
	(iii) The correlation between Y_1 and Y_2 is -0.25, the same as the correlation between X_1 and X_2. This illustrates the general point that re-scaling random variables (for example, by changing the units of measurement) changes their covariance but not their correlation.	**1** **1**
(b)	(i) $\rho_{12} = \rho_{23} = \rho, \quad \rho_{13} = \rho^2$	**1, 1**
	(ii) Y is normally distributed with	**1**
	$E(Y) = \frac{1}{3}(\mu + \mu + \alpha + \mu + 2\alpha) = \mu + \alpha$	**1, 1**
	$\text{Var}(Y) = (1/3 \quad 1/3 \quad 1/3)\sigma^2 \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix}\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$	**1**
	$= \frac{\sigma^2}{9}(3 + 4\rho + 2\rho^2)$	**1**
	(iii) $\underline{\bar{\mathbf{X}}}$ has a multivariate normal distribution with	**1**
	$E(\underline{\bar{\mathbf{X}}}) = \begin{pmatrix} \mu \\ \mu + \alpha \\ \mu + 2\alpha \end{pmatrix}, \quad \text{Cov}(\underline{\bar{\mathbf{X}}}) = \frac{1}{n}\begin{pmatrix} \sigma^2 & \sigma^2\rho & \sigma^2\rho^2 \\ \sigma^2\rho & \sigma^2 & \sigma^2\rho \\ \sigma^2\rho^2 & \sigma^2\rho & \sigma^2 \end{pmatrix}$	**1, 1**

Question 5

(i)	In this case, $f(x) = 1$ $(0 \le x \le 1)$ and $F(x) = x$ $(0 \le x \le 1)$. Setting $n = 3$, $i = 2$ and $j = 3$ gives	**1**
	$g(w,v) = \frac{3!}{1!0!0!}[w]^1.[v-w]^0.[1-v]^0 1.1 = 6w, \quad 0 \le w \le v \le 1$	**1, 1**
(ii)	$E(W^k V^m) = 6\int_0^1 w^{k+1}\int_w^1 v^m dv dw$	**1**
	$= \frac{6}{m+1}\int_0^1 w^{k+1}(1-w^{m+1})dw$	**1**
	$= \frac{6}{m+1}\left[\frac{w^{k+2}}{k+2} - \frac{w^{k+m+3}}{k+m+3}\right]_0^1$	**1**
	$= \frac{6}{(k+2)(k+m+3)}$	**1**
	$k = 1$, $m = 0$: $E(W) = 6/(3 \text{ x } 4) = 0.5$	**1**
	$k = 2$, $m = 0$: $E(W^2) = 6/(4 \text{ x } 5) = 0.3 \Rightarrow \text{Var}(W) = 0.3 - (0.5)^2 = 0.05$	**1, 1**
	$k = 0$, $m = 1$: $E(V) = 6/(2 \text{ x } 4) = 0.75$	**1**
	$k = 0$, $m = 2$: $E(V^2) = 6/(2 \text{ x } 5) = 0.6 \Rightarrow \text{Var}(V) = 0.6 - (0.75)^2 = 0.0375$	**1, 1**
	$k = 1$, $m = 1$: $E(WV) = 6/(3 \text{ x } 5) = 0.4$	**1**
	$\Rightarrow \text{Cov}(W, V) = 0.4 - (0.5) \text{ x } (0.75) = 0.025$	**1**
(iii)	The median is $W + \theta - ½$ and the maximum is $V + \theta - ½$.	**1**
	So the difference between the two estimates is $V - W$ (*or* $W - V$).	**1**
	$E(V - W) = E(V) - E(W) = 0.75 - 0.5 = 0.25$ (*or* $E(W - V) = -0.25$)	**1**
	$\text{Var}(V - W) = \text{Var}(V) + \text{Var}(W) - 2 \text{ x } \text{Cov}(W, V)$	**1**
	$= 0.0375 + 0.05 - 0.05 = 0.0375$	**1**

Question 6

(i)	$M_X(t) = E\left(e^{Xt}\right) = \sum_{x=0}^{m} e^{xt}\binom{m}{x}\theta^x(1-\theta)^{m-x}$	**1**
	$= \sum_{x=0}^{m}\binom{m}{x}(\theta e^t)^x(1-\theta)^{m-x}$	**1**
	$= (1-\theta+\theta e^t)^m$ [Binomial Theorem]	**1**
	$M'_X(t) = m(1-\theta+\theta e^t)^{m-1}\theta e^t$	**1**
	$\Rightarrow$ $E(X) = M'_X(0) = m\theta$	**1**
	$M''_X(t) = [m(1-\theta+\theta e^t)^{m-1}]\theta e^t + [m(m-1)(1-\theta+\theta e^t)^{m-2}\theta e^t]\theta e^t$	**1, 1**
	$\Rightarrow$ $E(X^2) = M''_X(0) = m\theta + m(m-1)\theta^2$	**1**
	$\text{Var}(X) = m\theta + m(m-1)\theta^2 - m^2\theta^2 = m\theta(1-\theta)$	**1**
(ii)	Each X_i has moment-generating function $M_i(t) = (1-\theta+\theta e^t)$.	**1**
	Let $S = X_1 + \ldots + X_n$. Since $X_1, \ldots, X_n$ are <u>independent</u> [*1 mark*], then	**1**
	$M_S(t) = M_1(t)\cdots M_n(t) = \left(1-\theta+\theta e^t\right)^n$	**1, 1**
	This is the moment-generating function of the Bi(n, θ) distribution. Using the <u>Uniqueness Property of moment-generating functions</u>, [*1 mark*] S ~ Bi(n, θ).	**1**
(iii)	The Central Limit Theorem: Suppose that X_1, ..., X_n is a sequence of <u>independent and identically-distributed</u> random variables, each with (finite) expected value μ and (finite) variance σ^2. For sufficiently large values of n,	**1**
	$\dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \sim \text{N}(0,1)$ approximately	**1**
	[*Give mark for correct statement in terms of* $\overline{X}$.]	
	In the context of part (ii), $\mu = \theta$ and $\sigma^2 = \theta(1-\theta)$ using the results proved in (i). So approximately, for large enough n,	**1**
	$\dfrac{S-n\theta}{\sqrt{n\theta(1-\theta)}} \sim \text{N}(0,1)$ approximately	**1**
	or $S \sim \text{N}\left(n\theta, n\theta(1-\theta)\right)$ approximately.	**1**
	It was proved in (ii) that S ~ Bi(n, θ). Therefore, the Bi(n, θ) distribution can be approximated by the $\text{N}\left(n\theta, n\theta(1-\theta)\right)$ distribution for large enough n.	**1**

Question 7

(a)	(i) The Tables provided give the following cumulative probabilities [*also give this mark if candidate works out point probabilities and uses them to calculate the correct cumulative probabilities*]:	
	x: 0, 1, 2, 3, 4, 5, … **$F_X(x)$**: 0.0821, 0.2873, 0.5438, 0.7576, 0.8912, 0.9580, …	**1**
	u_1 = 0.0885, so x_1 = 1 u_2 = 0.4096, so x_2 = 2	**1, 1**
	u_3 = 0.7370, so x_3 = 3 u_4 = 0.9384, so x_4 = 5	**1, 1**
	The pseudo-random variates are 1, 2, 3 and 5.	
	(ii) $F(x) = \int_3^x 10e^{-10(t-3)}dt = \left[-e^{-10(t-3)}\right]_3^x = 1 - e^{-10(x-3)}, \quad x > 3$	**1**
	So $u = F(x) \Leftrightarrow u = 1 - e^{-10(x-3)} \Leftrightarrow x = 3 - \frac{1}{10}\log_e(1-u)$	**1, 1**
	u_1 = 0.0885, so x_1 = 3.009 u_2 = 0.4096, so x_2 = 3.053	**½, ½**
	u_3 = 0.7370, so x_3 = 3.134 u_4 = 0.9384, so x_4 = 3.279	**½, ½**
	The pseudo-random variates are 3.009, 3.053, 3.134, 3.279.	
(b)	For full table of results, see next page. [*Give 1 mark for mapping six different digits to the possible outcomes on the die, 1 mark for discarding all occurrences of the other four digits, 1 mark for not starting till the first "6", 1 mark for applying the other conditions. If a candidate carrying out the correct general procedure makes minor errors, deduct 1 mark.*]	**4**
	This simulation required 58 random digits, giving 33 valid rolls of a die.	**1, 1**
	Repeat the simulation many times (possibly 1000 or 10000). Count the number of rolls of the die required each time. Obtain an appropriate interval based on the distribution of the number of rolls required (for example, mean +/- 1.96s.d., or 2.5th to 97.5th percentile of the sample distribution).	**1** **1** **2**

Question 7

#	Digit	Outcome	#	Digit	Outcome
1	5	-	31	0	Not Roll
2	2	-	32	0	Not Roll
3	1	-	33	6	-
4	0	Not Roll	34	2	**Antenna 1**
5	9	Not Roll	35	6	-
6	0	Not Roll	36	6	-
7	0	Not Roll	37	2	**Antenna 2**
8	9	Not Roll	38	0	Not Roll
9	6	**Body**	39	6	-
10	8	Not Roll	40	8	Not Roll
11	6	-	41	7	Not Roll
12	8	Not Roll	42	0	Not Roll
13	0	Not Roll	43	5	-
14	8	Not Roll	44	4	-
15	9	Not Roll	45	9	Not Roll
16	8	Not Roll	46	9	Not Roll
17	6	-	47	6	-
18	3	**Leg 1**	48	7	Not Roll
19	6	-	49	5	-
20	4	**Tail**	50	6	-
21	2	-	51	5	-
22	8	Not Roll	52	3	**Leg 3**
23	8	Not Roll	53	9	Not Roll
24	4	-	54	4	-
25	5	**Head**	55	4	-
26	4	-	56	3	**Leg 4**
27	1	**Eye 1**	57	5	-
28	9	Not Roll	58	1	**Eye 2**
29	3	**Leg 2**			
30	9	Not Roll			

Question 8

(i)	$$U = 1 - X,\ V = \frac{Y}{1-X}$$	
	means $X = 1 - U,\ Y = UV$	**1, 1**
	$$J = \begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ V & U \end{vmatrix} = -U$$	**4@½,1**
	Hence, on the range space $\{(u, v): 0 \le u \le 1;\ 0 \le v \le 1\}$,	**1**
	$$g(u,v) = u\, f(1-u,\ uv)$$	**1**
	$$= u \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}(1-u)^{\alpha-1}(uv)^{\beta-1}(u-uv)^{\gamma-1}$$	**1**
	$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\beta+\gamma-1}(1-u)^{\alpha-1}v^{\beta-1}(1-v)^{\gamma-1}$$	**1**
(ii)	The joint p.d.f. factorises as the product of a function of u alone and a function of v alone. Their joint range space is a Cartesian product (or 'rectangular space'). Using the Factorisation Theorem, therefore, U and V are independent. The marginal p.d.f.'s are	**1** **1**
	$$g(u) \propto u^{\beta+\gamma-1}(1-u)^{\alpha-1},\ 0 \le u \le 1$$	**1**
	so $g(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\beta+\gamma)\Gamma(\alpha)} u^{\beta+\gamma-1}(1-u)^{\alpha-1},\ 0 \le u \le 1,$	**1**
	i.e. $U \sim \text{Be}(\beta+\gamma,\ \alpha)$ [*Give full marks to a candidate who carries out these last two steps in the reverse order.*]	**1**
	and $g(v) \propto v^{\beta-1}(1-v)^{\gamma-1},\ 0 \le v \le 1$	**1**
	so $g(v) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)} v^{\beta-1}(1-v)^{\gamma-1},\ 0 \le v \le 1,$	**1**
	i.e. $V \sim \text{Be}(\beta,\ \gamma)$ [*Give full marks to a candidate who carries out these last two steps in the reverse order.*]	**1**
(iii)	$X = 1 - U$, so $X \sim \text{Be}(\alpha,\ \beta+\gamma)$	**1**
	$\therefore$ $Y \sim \text{Be}(\beta,\ \alpha+\gamma)$ and $Z \sim \text{Be}(\gamma,\ \alpha+\beta)$, by symmetry	**1, 1**