EXAMINATIONS OF THE HONG KONG STATISTICAL SOCIETY

HIGHER CERTIFICATE IN STATISTICS, 2013

MODULE 2 : Probability models

Time allowed: One and a half hours

Candidates should answer **THREE** *questions.*

Each question carries 20 marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use calculators in accordance with the regulations published in the Society's "Guide to Examinations" (document Ex1).

The notation log *denotes logarithm to base* ***e***.
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^nC_r$.

This examination paper consists of 8 printed pages.
This front cover is page 1.
Question 1 starts on page 2.

There are 4 questions altogether in the paper.

1. (i) In Newtopia, the weather on any day is dry with probability $\frac{2}{3}$ and wet with probability $\frac{1}{3}$, the weather on different days being independent.

(a) Find the probability that the next three days are dry.

(2)

(b) Find the probability that exactly two of the next three days are wet.

(3)

(c) A Newtopian resident walks his dog with probability 0.9 when it is dry but with probability 0.6 when it is wet. If it is known that he walked his dog last Tuesday, what is the probability that last Tuesday was dry in Newtopia?

(5)

(ii) Suppose now that the weather on different days is not independent but that

$$P(\text{next day is dry} \mid \text{today is dry}) = 0.8$$

and $$P(\text{next day is wet} \mid \text{today is wet}) = 0.6.$$

(a) Given that today is dry, what is the probability that the next three days are dry?

(2)

(b) Given that today is wet, what is the probability that exactly two of the next three days are wet?

(4)

(c) Let p denote the overall probability that any day is dry. Explain clearly why p must satisfy the equation

$$0.8p + 0.4(1 - p) = p,$$

and deduce the value of p.

(4)

2. The random variable X has the exponential distribution with probability density function

$$f(x) = \lambda e^{-\lambda x}, \; x > 0, \; \lambda > 0.$$

(a) (i) Show that $E(X) = \dfrac{1}{\lambda}$.

(3)

(ii) Show that $P(X > a) = e^{-\lambda a}$ for any $a > 0$. Deduce the median of X.

(4)

(iii) For any $b > 0$, find $P(X > a + b \mid X > a)$ and comment on this result.

(3)

(b) Now consider the case where $\lambda = 1$.

(i) Sketch the graph of $f(x)$.

(3)

(ii) State with a reason whether the distribution of X is positively or negatively skew.

(2)

(iii) Write down the mode of the distribution of X, and find the value of k such that *Mean – Mode* = k(*Mean – Median*).

(3)

(iv) A student has read that, for many distributions,

- the skewness is positive if the mean is greater than the median
- the value of k is about 3.

Comment on the truth of each of these statements for the distribution of X.

(2)

3. The random variable X has the binomial distribution with probability mass function

$$p(x) = \frac{6!}{x!(6-x)!}\left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{6-x}, \quad x = 0, 1, 2, \ldots, 6.$$

(i) Derive $E(X)$ and $\mathrm{Var}(X)$.

(7)

(ii) Find $P(X \leq 1)$

(a) exactly,

(b) by using a Normal approximation with a suitable continuity correction.

State circumstances under which a Normal approximation to the binomial distribution might be useful, and comment on your results.

(8)

(iii) Let $\bar{X}$ be the mean of a random sample of size 400 taken from the distribution of X. Calculate $\mathrm{Var}(\bar{X})$, and use a Normal approximation to the distribution of $\bar{X}$ to find $P(2.35 < \bar{X} \leq 2.45)$. State with a reason whether or not you would expect your answer to be a good approximation to the exact probability.

(5)

4. (a) The random variable X has the Poisson distribution with mean 1.5, so that

$$P(X = x) = e^{-1.5}\frac{1.5^x}{x!}, \quad x = 0, 1, 2, \ldots .$$

Draw a graph of $P(X = x)$ versus x, showing all probabilities greater than 0.02, and write down the mean, mode and variance of this distribution.

(7)

(b) Assume that, for any $t > 0$, the number N of incoming telephone calls to a 24-hour international call centre in a time t minutes follows a Poisson distribution with mean $3t$.

(i) Find the probability that exactly 2 calls arrive in the next minute.

(2)

(ii) Find the probability that no calls arrive in the next $1\frac{1}{2}$ minutes.

(2)

(iii) State the exact distribution of N in the case $t = 60$. Staffing levels at the call centre are based on the assumption that not more than 200 calls will be made in an hour. Use a suitable approximation to the distribution of N to calculate the approximate probability that this assumption is violated in a given hour.

(6)

(iv) Comment critically on the assumption made at the beginning of part (b).

(3)

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