EXAMINATIONS OF THE HONG KONG STATISTICAL SOCIETY

HIGHER CERTIFICATE IN STATISTICS, 2012

MODULE 5 : Further probability and inference

Time allowed: One and a half hours

Candidates should answer **THREE** *questions.*

Each question carries 20 marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use calculators in accordance with the regulations published in the Society's "Guide to Examinations" (document Ex1).

The notation log *denotes logarithm to base* ***e****.*
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^nC_r$.

This examination paper consists of 4 printed pages, **each printed on one side only**.
This front cover is page 1.
Question 1 starts on page 2.

There are 4 questions altogether in the paper.

1. The continuous random variables X and Y are jointly distributed with joint probability density function

$$f(x, y) = \begin{cases} xe^{-(1+y)x} & x > 0,\ y > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the discrete random variables W and V are defined by

$$W = \begin{cases} 0 & \text{if } X < 1, \\ 1 & \text{if } X \geq 1, \end{cases} \qquad V = \begin{cases} 0 & \text{if } Y < 1, \\ 1 & \text{if } Y \geq 1. \end{cases}$$

(i) Find the marginal probability density functions of X and Y.

(6)

(ii) Find the conditional density function $f(y \mid x)$ for $x > 0$ and hence evaluate $E(Y \mid X = x)$.

(5)

(iii) Find $P(W = 1)$, $P(V = 1)$ and $P(W = V = 1)$.

(5)

(iv) Find $\text{Cov}(W, V)$.

(4)

2. (a) Suppose that X is a discrete random variable which can only take non-negative integer values (i.e. 0, 1, 2, …). Define the probability generating function $\pi_X(t)$ of X.

(2)

(b) The discrete random variable Y has probability distribution

$$P(Y = k) = \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} \quad \text{for } k = 1, 2, 3, \ldots ,$$

where λ is a positive parameter.

(i) Show that the probability generating function of Y is $te^{\lambda(t-1)}$.

(4)

(ii) Use the probability generating function of Y to evaluate the mean and variance of Y. Outline how you would find $E(Y^3)$.

(8)

(iii) Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent random variables, each with the same distribution as Y, and that $W = \sum_{i=1}^{n} Y_i$. Find the probability generating function of W and hence find $P(W = k)$ for $k = 0, 1, 2, 3, \ldots$.

(6)

Turn over

3. (a) What is meant by a *maximum likelihood estimator*? What good properties do maximum likelihood estimators possess, assuming that the necessary regularity conditions hold?

(8)

(b) The probability of obtaining heads when a coin is tossed is p, where $0 < p < 1$ and p is unknown. A game consists of two independent tosses of the coin, with the player winning if there is exactly one head. In 100 independent games, the player wins 30 times.

(i) Show that the likelihood of p is proportional to

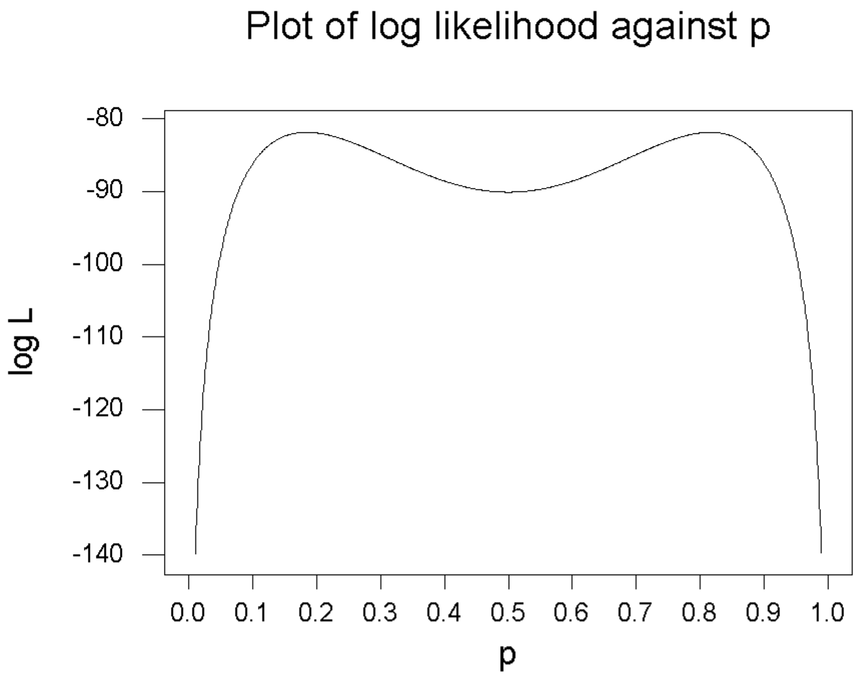
$$p^{30}(1-p)^{30}\left(1-2p(1-p)\right)^{70}$$

for $0 < p < 1$.

(5)

(ii) The plot of the logarithm of the likelihood shown below indicates that there are two values of p that give the same maximum value for the likelihood. Explain why this happens and find an equation satisfied by these values. (Do not attempt to solve this equation.)

(7)



Turn over

4. Random variables X_1 and X_2 are independent, each having a Normal distribution with mean and variance both equal to $\theta > 0$, an unknown parameter. It is required to use X_1 and X_2 to estimate θ.

(i) Find the method of moments estimator $\hat{\theta}$ of θ based on the first sample moment. Show that $\hat{\theta}$ is unbiased and find its variance.

(4)

(ii) The random variable $Y = X_1 - X_2$. State the distribution of Y.

(1)

(iii) Show that $\hat{\theta}$ and Y are uncorrelated.

(3)

(iv) Another unbiased estimator of θ is $\tilde{\theta} = kY^2$, where k is a constant. Find the value of k.

(4)

(v) Show that $E(Y^4) = 12\theta^2$.

(4)

[You may use the result that the moment generating function of a Normal distribution with mean μ and variance σ^2 is $\exp(\mu t + \frac{1}{2}\sigma^2 t^2)$.]

(vi) Use part (v) to find the variance of $\tilde{\theta}$, and hence the efficiency of $\hat{\theta}$ relative to $\tilde{\theta}$.

(4)