THE ROYAL STATISTICAL SOCIETY

2009 EXAMINATIONS – SOLUTIONS

HIGHER CERTIFICATE

MODULE 5

FURTHER PROBABILITY AND INFERENCE

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Note. In accordance with the convention used in the Society's examination papers, the notation log denotes logarithm to base *e*. Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Higher Certificate, Module 5, 2009. Question 1
(Solution continues on next page)

$$f(x,y)=\begin{cases}\dfrac{1}{3\log 2}\left(\dfrac{x}{y}+\dfrac{y}{x}\right) & 1\le x\le 2,\ 1\le y\le 2\\ 0 & \text{otherwise}\end{cases}.$$

(i) $$f(x)=\int_1^2\frac{1}{3\log 2}\left(\frac{x}{y}+\frac{y}{x}\right)dy=\frac{1}{3\log 2}\left[x\log y+\frac{y^2}{2x}\right]_{y=1}^{y=2}$$

$$=\frac{1}{3\log 2}\left(\left[x\log 2-0\right]+\frac{1}{2x}\left[4-1\right]\right)=\frac{x}{3}+\frac{1}{2x\log 2}\qquad(\text{for } 1\le x\le 2).$$

(ii) $$E(X)=\int_1^2 x\,f(x)\,dx=\int_1^2\left(\frac{x^2}{3}+\frac{1}{2\log 2}\right)dx=\left[\frac{x^3}{9}+\frac{x}{2\log 2}\right]_1^2$$

$$=\frac{7}{9}+\frac{1}{2\log 2}=1.4991252\approx 1.4991.$$

$$E\left(X^2\right)=\int_1^2 x^2 f(x)\,dx=\int_1^2\left(\frac{x^3}{3}+\frac{x}{2\log 2}\right)dx=\left[\frac{x^4}{12}+\frac{x^2}{4\log 2}\right]_1^2$$

$$=\frac{15}{12}+\frac{3}{4\log 2}=2.3320213.$$

$$\therefore\ \text{Var}(X)=2.3320213-(1.4991252)^2\approx 0.0847.$$

(iii) $$E(XY)=\int_1^2\int_1^2 xy\,f(x,y)\,dydx=\int_1^2\int_1^2\frac{1}{3\log 2}\left(x^2+y^2\right)dydx$$

$$=\frac{1}{3\log 2}\int_1^2\left[x^2y+\frac{y^3}{3}\right]_{y=1}^{y=2}dx=\frac{1}{3\log 2}\int_1^2\left(x^2+\frac{7}{3}\right)dx$$

$$=\frac{1}{3\log 2}\left[\frac{x^3}{3}+\frac{7x}{3}\right]_1^2$$

$$=\frac{1}{3\log 2}\left(\frac{7}{3}+\frac{7}{3}\right)=\frac{14}{9\log 2}=2.2441923\approx 2.2442.$$

(iv) By symmetry, $E(Y) = 1.4991252$.

$$\therefore \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 2.2441923 - 1.4991252^2$$

$$= -0.003184079$$

(v) $$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\left(\frac{x}{y} + \frac{y}{x}\right)}{3\log 2\left(\frac{x}{3} + \frac{1}{2x\log 2}\right)} \qquad (\text{for } 1 \le x \le 2,\ 1 \le y \le 2)$$

$$\therefore P(Y < 1.5|X = 1) = \int_1^{1.5} f(y|1)\,dy = \frac{1}{3\log 2\left(\frac{1}{3} + \frac{1}{2\log 2}\right)}\int_1^{1.5}\left(\frac{1}{y} + y\right)dy$$

$$= \frac{1}{\log 2 + \frac{3}{2}}\left[\log y + \frac{y^2}{2}\right]_1^{1.5} = \frac{\log 1.5 + \frac{5}{8}}{\log 2 + \frac{3}{2}}$$

$$= \frac{1.030465}{2.1931472} \approx 0.4699.$$

(i) $$\frac{dm(t)}{dt} = -2\times\left(-\frac{k}{2}\right)(1-2t)^{-\frac{k}{2}-1} = k(1-2t)^{-\frac{k}{2}-1}$$

$$\therefore E(X) = \left.\frac{dm(t)}{dt}\right|_{t=0} = k\ .$$

$$\frac{d^2m(t)}{dt^2} = -2k\times\left(-\frac{k}{2}-1\right)(1-2t)^{-\frac{k}{2}-2} = k(k+2)(1-2t)^{-\frac{k}{2}-2}$$

$$\therefore E(X^2) = \left.\frac{d^2m(t)}{dt^2}\right|_{t=0} = k(k+2).$$

$$\therefore \mathrm{Var}(X) = E(X^2) - (E(X))^2 = k^2 + 2k - k^2 = 2k\ .$$

(ii) $$m(t) = \int_0^\infty e^{tx} f(x)\,dx = \frac{1}{4}\int_0^\infty x e^{-x\left(\frac{1}{2}-t\right)}dx$$

$$= \frac{1}{4}\left\{\left[x\frac{e^{-x\left(\frac{1}{2}-t\right)}}{-\left(\frac{1}{2}-t\right)}\right]_0^\infty - \int_0^\infty \frac{e^{-x\left(\frac{1}{2}-t\right)}}{-\left(\frac{1}{2}-t\right)}dx\right\}$$

$$= \frac{1}{4}\left[-\frac{e^{-x\left(\frac{1}{2}-t\right)}}{\left(\frac{1}{2}-t\right)^2}\right]_0^\infty = \frac{1}{4}\times\frac{1}{\left(\frac{1}{2}-t\right)^2} = (1-2t)^{-2}, \quad \text{as required.}$$

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(iii) $m_{Y_i}(t) = (1-2t)^{-\frac{1}{2}} \qquad (i = 1, 2, ..., n; \;\; t < \tfrac{1}{2})$.

By the convolution theorem,

$$m_V(t) = \prod_{i=1}^{n} m_{Y_i}(t) = (1-2t)^{-\frac{n}{2}}, \quad \text{and this is the mgf of } \chi_n^2.$$

Therefore, by the 1:1 correspondence between mgfs and distributions, $V \sim \chi_n^2$.

(iv) For $n = 300$, we have $V \sim \chi_{300}^2$.

By part (i), $E(V) = 300$ and $\text{Var}(V) = 2 \times 300 = 600$.

By the central limit theorem, since V is the sum of a large number of random variables (independent identically distributed, finite variance), V has a Normal distribution, $V \sim \text{N}(300, 600)$, approximately.

$$\therefore P(V \le 310) \approx \Phi\left(\frac{310-300}{\sqrt{600}}\right) = \Phi(0.4082) = 0.6584.$$

<u>Higher Certificate, Module 5, 2009. Question 3</u>

(i) $$1=\sum_{k=1}^{\infty}P(X=k)=Ce^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}=Ce^{-\lambda}\left(\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}-1\right)$$

$$=Ce^{-\lambda}\left(e^{\lambda}-1\right)=C\left(1-e^{-\lambda}\right)$$

$\therefore C=\left(1-e^{-\lambda}\right)^{-1}$ as required.

(ii) The likelihood is $L(\lambda)=\prod_{i=1}^{n}P(X_i=x_i)=\dfrac{\left(1-e^{-\lambda}\right)^{-n}e^{-n\lambda}\lambda^{\Sigma xi}}{\prod x_i!}$.

$\therefore$ Log likelihood is $\ell(\lambda)=n\log\left(1-e^{-\lambda}\right)-n\lambda+\Sigma x_i\log\lambda-\log\left(\Pi x_i!\right)$.

The maximum likelihood estimator $\hat{\lambda}$ satisfies the equation $\dfrac{d\ell}{d\lambda}=0$, i.e. it satisfies $\dfrac{-ne^{-\lambda}}{1-e^{-\lambda}}-n+\dfrac{\Sigma x_i}{\lambda}=0$.

(iii) $$\frac{d^2\ell}{d\lambda^2}=\frac{\left(1-e^{-\lambda}\right)ne^{-\lambda}-\left(-ne^{-\lambda}\right)e^{-\lambda}}{\left(1-e^{-\lambda}\right)^2}-\frac{\Sigma x_i}{\lambda^2}=\frac{ne^{-\lambda}}{\left(1-e^{-\lambda}\right)^2}-\frac{\Sigma x_i}{\lambda^2}.$$

Now, $E(X)=\dfrac{1}{\left(1-e^{-\lambda}\right)}\displaystyle\sum_{k=1}^{\infty}k\frac{e^{-\lambda}\lambda^k}{k!}=\frac{\lambda}{1-e^{-\lambda}}$.

$$\therefore E\left(-\frac{d^2\ell}{d\lambda^2}\right)=-\frac{ne^{-\lambda}}{\left(1-e^{-\lambda}\right)^2}+\frac{n\lambda}{\lambda^2\left(1-e^{-\lambda}\right)}=\frac{n\left(1-e^{-\lambda}-\lambda e^{-\lambda}\right)}{\lambda\left(1-e^{-\lambda}\right)^2}.$$

$$\therefore \operatorname{Var}\left(\hat{\lambda}\right)\approx\frac{\hat{\lambda}\left(1-e^{-\hat{\lambda}}\right)^2}{n\left(1-e^{-\hat{\lambda}}-\hat{\lambda}e^{-\hat{\lambda}}\right)}.$$

Solution continued on next page

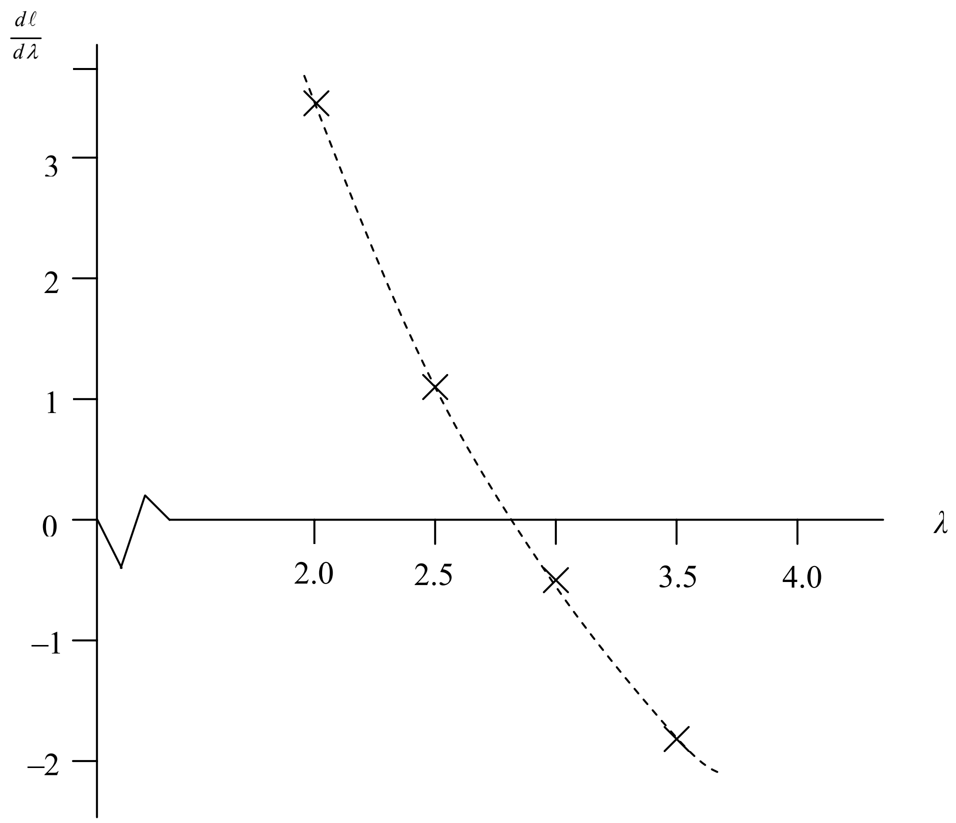
(iv) For $n = 10$ and $\Sigma X_i = 30$, we have $\frac{d\ell}{d\lambda} = \frac{-10e^{-\lambda}}{1-e^{-\lambda}} - 10 + \frac{30}{\lambda}$. The values of this at the stated points are:

λ	2.0	2.5	3.0	3.5
$d\ell / d\lambda$	3.43	1.11	–0.52	–1.74

See the graph below. From the graph, $\frac{d\ell}{d\lambda} = 0$ at approximately $\lambda = 2.8$.

So 2.8 is (approximately) the required value of the maximum likelihood estimator (consideration of the gradient of $d\ell / d\lambda$ shows that this is indeed a maximum).

Derivative of log(likelihood) versus λ



Higher Certificate, Module 5, 2008. Question 4

(i) Using the given probability generating function, $\dfrac{d\pi(t)}{dt} = \dfrac{p(1-p)}{\left(1-(1-p)t\right)^2}$.

$$\therefore E(Y) = \left.\frac{d\pi(t)}{dt}\right|_{t=1} = \frac{p(1-p)}{p^2} = \frac{1-p}{p}.$$

Also, $\dfrac{d^2\pi}{dt^2} = \dfrac{2p(1-p)^2}{\left(1-(1-p)t\right)^3}$.

$$\therefore \left.\frac{d^2\pi}{dt^2}\right|_{t=1} = \frac{2p(1-p)^2}{p^3} = \frac{2(1-p)^2}{p^2}.$$

$$\therefore \text{Var}(Y) = \frac{2(1-p)^2}{p^2} + \frac{1-p}{p} - \left(\frac{1-p}{p}\right)^2 = \frac{1-p}{p^2}.$$

(ii) $E(\bar{Y}) = \dfrac{1}{n}\sum_{i=1}^{n} E(Y_i) = \dfrac{1-p}{p} = \dfrac{1}{p} - 1$, so $\bar{Y}$ is a biased estimator of $\dfrac{1}{p}$.

But $E(\bar{Y}+1) = \dfrac{1}{p}$, so $\bar{Y}+1$ is an unbiased estimator of $\dfrac{1}{p}$.

$$\text{Var}(\bar{Y}+1) = \text{Var}(\bar{Y}) = \frac{1}{n^2}\sum \text{Var}(Y_i) = \frac{n(1-p)}{n^2p^2} = \frac{1-p}{np^2}$$

and this $\to 0$ as $n \to \infty$.

Since $\bar{Y}+1$ is unbiased for $\dfrac{1}{p}$ (for all n) and its variance tends to zero as $n \to \infty$, $\bar{Y}+1$ is a consistent estimator of $\dfrac{1}{p}$.

Solution continued on next page

(iii) The method of moments estimator of p is the solution, $\hat{p}$ say, of $\overline{Y} = E(\overline{Y})$, i.e. we have $\overline{Y} = \frac{1}{\hat{p}} - 1$. $\therefore \frac{1}{\hat{p}} = \overline{Y} + 1$, i.e. $\hat{p} = \frac{1}{\overline{Y} + 1}$.

(iv) We have $P(Y_i = 0) = (1-p)^0 p = p$. So the distribution of W is B(n, p).

$\therefore E(W) = np$, and so W/n is an unbiased estimator of p.

$$\mathrm{Var}\left(\frac{W}{n}\right) = \frac{1}{n^2}\mathrm{Var}(W) = \frac{1}{n^2}np(1-p) = \frac{p(1-p)}{n}.$$