THE ROYAL STATISTICAL SOCIETY

2009 EXAMINATIONS – SOLUTIONS

GRADUATE DIPLOMA

MODULAR FORMAT

MODULE 1

PROBABILITY DISTRIBUTIONS

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Graduate Diploma, Module 1, 2009. Question 1

(i) $$P(E_1|A)=\frac{P(A|E_1)P(E_1)}{\sum_{i=1}^{n}P(A|E_i)P(E_i)}.$$

(ii) Define the following events, which partition the sample space.

E_1 : it was a £20 note

E_2 : it was a £10 note

(A) As all thirty notes are mixed up at random, each is equally likely to be the one that Brian used, so $P(E_1)$ is simply 20/30 = 2/3.

(B) Now define event A : Anna says it was a £10 note. We seek $P(E_1|A)$.

We have $P(A|E_1=0.1)$, $P(A|E_2=0.9)$ and $P(E_1)=2/3$, $P(E_2)=1/3$.

$$\therefore P(E_1|A)=\frac{0.1\times(2/3)}{[0.1\times(2/3)]+[0.9\times(1/3)]}=\frac{2}{2+9}=\frac{2}{11}.$$

(C) We now begin with $P(E_1)=2/11$ and $P(E_2)=9/11$. We define event B : Brian says it was a £20 note.

We have $P(B|E_1=0.8)$, $P(B|E_2=0.3)$. So now

$$P(E_1|B)=\frac{0.8\times(2/11)}{[0.8\times(2/11)]+[0.3\times(9/11)]}=\frac{16}{16+27}=\frac{16}{43}.$$

Graduate Diploma, Module 1, 2009. Question 2

Part (i)

We have, as a general result, $P(A) = P(A \cap B) + P(A \cap \overline{B})$.

Also, by independence, $P(A \cap B) = P(A)P(B)$.

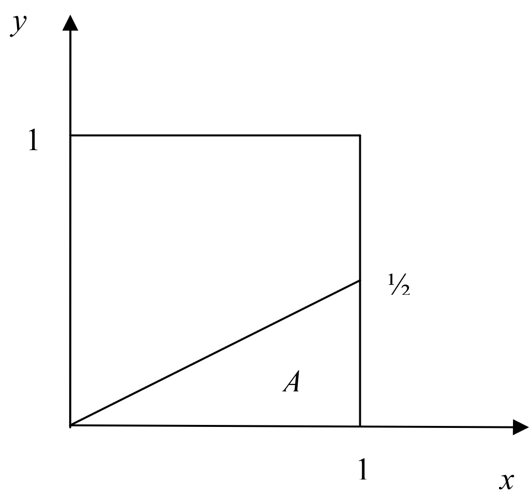
$$\therefore P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - (P(A)P(B))$$
$$= P(A)\{1 - P(B)\} = P(A)P(\overline{B}),$$

showing that A and $\overline{B}$ are independent.

A similar argument starting from $P(\overline{A}) = P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B})$ shows that $\overline{A}$ and $\overline{B}$ are independent.

Part (ii)

(a)



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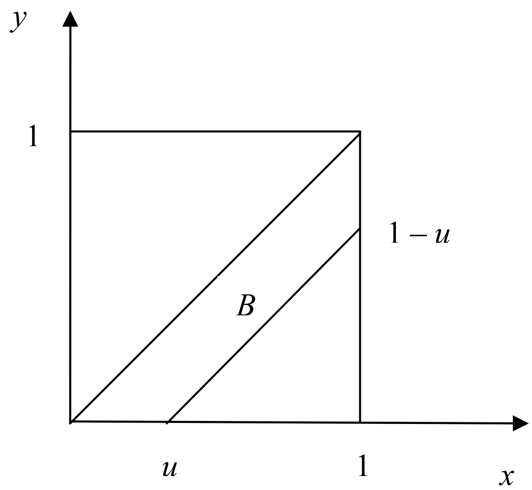
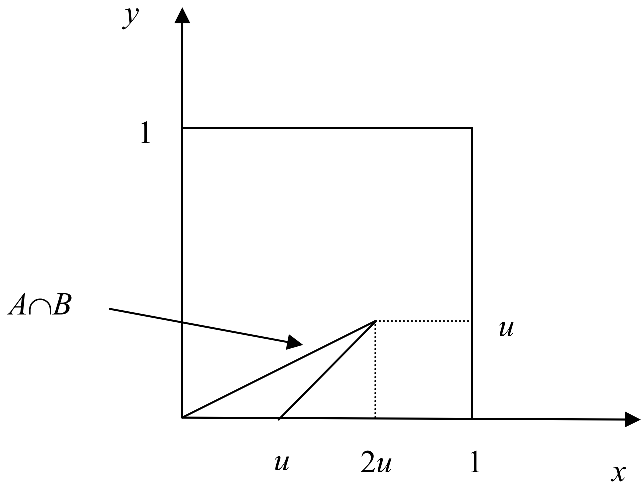




(b) Simple geometry gives $P(A) = ¼$ and $P(B) = 0.5 – 0.5(1 – u)^2 = u – (u^2/2)$. Also, from the formula for the area of a triangle, $P\left(A \cap B\right) = (1/2)u.u = u^2/2$.

$\therefore$ independence requires $\frac{u^2}{2} = \frac{1}{4} \times \left(u - \frac{u^2}{2} \right)$, i.e. $2u^2 = u – (u^2/2)$. As $u \neq 0$, the only solution of this is $u = 2/5$.

(c) With $u = 2/5$, we have $P(B) = 8/25$.

$\therefore P\left(\bar{A} \cap \bar{B}\right) = P\left(\bar{A}\right)P\left(\bar{B}\right) = (3/4)(17/25) = 51/100$.

Graduate Diploma, Module 1, 2009. Question 3

$X \sim$ Poisson(λ) $\quad Y \sim$ Poisson(μ) $\quad X$ and Y are independent

(i) For any $n = 0, 1, 2, \ldots$ we have

$$P(X+Y=n) = \sum_{k=0}^{n} P(X+Y=n \cap X=k) = \sum_{k=0}^{n} P(Y=n-k \cap X=k)$$

$$= \sum_{k=0}^{n} \frac{e^{-\mu}\mu^{n-k}}{(n-k)!} \cdot \frac{e^{-\lambda}\lambda^{k}}{k!} = \frac{e^{-\lambda-\mu}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda^{k} \mu^{n-k} = \frac{e^{-\lambda-\mu}(\lambda+\mu)^{n}}{n!},$$

so that $X + Y$ follows a Poisson distribution (with mean $\lambda + \mu$).

[Note. An argument based on probability generating functions was also acceptable in the examination, all required results being derived as appropriate.]

(ii) $$P(X=k \mid X+Y=n) = \frac{P(X=k \cap X+Y=n)}{P(X+Y=n)} = \frac{P(X=k \cap Y=n-k)}{P(X+Y=n)}$$

$$= \frac{\dfrac{e^{-\lambda}\lambda^{k}}{k!} \times \dfrac{e^{-\mu}\mu^{n-k}}{(n-k)!}}{\dfrac{e^{-\lambda-\mu}(\lambda+\mu)^{n}}{n!}} = \binom{n}{k} \frac{\lambda^{k}\mu^{n-k}}{(\lambda+\mu)^{n}} = \binom{n}{k} p^{k} q^{n-k}$$

where $p = \dfrac{\lambda}{\lambda+\mu}$ and $q = \dfrac{\mu}{\lambda+\mu}$ $(=1-p)$.

The distribution is binomial with parameters n and p.

(iii) Assume that the urgent and standard calls follow Poisson distributions with means 4 and 12 respectively over this 30 minute period. Given that there are 10 calls altogether, part (ii) shows that the distribution of the number of urgent calls is B(10, p) where $p = 4/(4 + 12) = ¼$. The probability that at most two are urgent is therefore simply $P(U \leq 2)$ where $U \sim$ B(10, ¼) which (from cumulative tables or by direct calculation) is 0.5256, i.e. 0.53 to two significant figures.

Graduate Diploma, Module 1, 2009. Question 4

(i) Write $f(x) = \log(x)$, so that the derivative is $f'(x) = 1/x$. With $Y = \log(X)$, we have the Taylor series expansion

$$Y \approx \log(\mu) + (X-\mu)f'(\mu) = \log(\mu) + \frac{X-\mu}{\mu},$$

and therefore $\left(Y - \log(\mu)\right)^2 \approx \dfrac{(X-\mu)^2}{\mu^2}$.

Taking expectations, and assuming that $E(\log(X)) \approx \log(\mu)$, this gives

$$\text{Var}(Y) \approx \frac{\text{Var}(X)}{\mu^2}.$$

So, as we are given that Var(X) is proportional to μ^2, Var(Y) is approximately constant.

(ii) $$E\left(X^k\right) = \int_0^\infty x^k \frac{\nu^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\nu x} dx = \int_0^\infty \frac{\nu^\alpha}{\Gamma(\alpha)} x^{k+\alpha-1} e^{-\nu x} dx$$

$$= \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \frac{1}{\nu^k} \int_0^\infty \{\text{pdf of } \Gamma(k+\alpha, \nu)\}\, dx = \frac{\Gamma(k+\alpha)}{\nu^k \Gamma(\alpha)}.$$

[Alternatively, this follows from the second integral above using substitution and the given result $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.]

In particular, this gives $E(X) = \dfrac{\Gamma(1+\alpha)}{\nu\,\Gamma(\alpha)} = \dfrac{\alpha\,\Gamma(\alpha)}{\nu\,\Gamma(\alpha)} = \dfrac{\alpha}{\nu}$ and, similarly, $E\left(X^2\right) = \dfrac{\alpha(\alpha+1)}{\nu^2}$.

$$\therefore \text{Var}(X) = \frac{\alpha(\alpha+1)}{\nu^2} - \frac{\alpha^2}{\nu^2} = \frac{\alpha}{\nu^2}.$$

Thus we have $\text{Var}(X) = [E(X)]^2/\alpha$, so the variance of X is proportional to the square of its mean as ν varies, as required.

Graduate Diploma, Module 1, 2009. Question 5

(i) We have $1 = \int_0^1\left(\int_x^1 kxy\,dy\right)dx = k\int_0^1 x\frac{1-x^2}{2}dx = k\left[\frac{x^2}{4}-\frac{x^4}{8}\right]_0^1 = \frac{k}{8}$,

and therefore $k = 8$.

Marginal probability density of X is given by

$$\int_x^1 8xy\,dy = 8x\frac{1-x^2}{2} = 4x-4x^3,\ \text{for } 0<x<1.$$

Marginal probability density of Y is given by

$$\int_0^y 8xy\,dx = 8y\frac{y^2}{2} = 4y^3,\ \text{for } 0<y<1.$$

X and Y are not independent. This follows because the joint density ($8xy$) is not the product of the two marginal densities, or by noting that the region on which the joint density is non-zero has a side ($x = y$) that is not parallel to an axis.

(ii) $$E\left(X^rY^s\right) = 8\int_0^1 x^{r+1}\left(\int_x^1 y^{s+1}dy\right)dx = \frac{8}{s+2}\int_0^1 x^{r+1}\left[1-x^{s+2}\right]dx$$

$$= \frac{8}{s+2}\left[\frac{1}{r+2}-\frac{1}{r+s+4}\right] = \frac{8}{(r+2)(r+s+4)}.$$

This applies for all non-negative integers r and s. So we have

$E(X) = 8/15$, $E(Y) = 4/5$, $E(X^2) = 1/3$, $E(Y^2) = 2/3$, $E(XY) = 4/9$.

Thus $\text{Var}(X) = (1/3) - (8/15)^2 = 11/225$ and $\text{Var}(Y) = (2/3) - (4/5)^2 = 2/75$.

$\therefore \text{Cov}(X,Y) = \frac{4}{9}-\frac{32}{75} = \frac{4}{225}$ and $\text{Corr}(X,Y) = \frac{4/225}{\sqrt{(11/225)(2/75)}} = \frac{4}{\sqrt{66}}$.

Graduate Diploma, Module 1, 2009. Question 6

(i) Let X_1, X_2, X_3, ... be independent identically distributed random variables, all with mean μ and variance σ^2 (> 0).

Write $Z_n = \dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}$.

Then $P(Z_n \le t) \to P(Z \le t)$ as $n \to \infty$, where Z is the standard Normal random variable.

(ii) When X has a continuous distribution, the values of $P(X < k)$ and $P(X \le k)$ are the same, whatever the value of k. But when X takes integer values only, including the value k, these probabilities differ. The continuity correction endeavours to take account of this by noting that, in the case where X takes integer values only, $P(X \le k) = P(X \le k + 0.5)$. Hence, when passing from the exact discrete case to an approximating continuous distribution, replacing k by $k + 0.5$ is expected to compensate.

(iii) In both (a) and (b), we rely on n being large enough to invoked the asymptotic result of part (i) as a suitable approximation.

(a) Take X_i as a Bernoulli variable with parameter p, i.e. $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$. Then $Y = X_1 + X_2 + ... + X_n$ has the required binomial distribution with parameters n and p, and has mean np and variance $np(1 - p)$.

Now write $a_n = \dfrac{a - np}{\sqrt{np(1-p)}}$ and $b_n = \dfrac{b - np}{\sqrt{np(1-p)}}$. Then we have

$$P(a \le Y \le b) = P(a_n \le Z_n \le b_n)$$

where Z_n is as in part (i), and this is approximated by $P(a_n \le Z \le b_n)$ where Z is the standard Normal random variable.

(b) Here take X_i as a Poisson variable with parameter λ/n where n is chosen to be close to λ (e.g. take n as the integral part of λ), so that $Y = X_1 + X_2 + ... + X_n$ has the required Poisson distribution with parameter λ.

Write $a_\lambda = (a - \lambda)/\sqrt{\lambda}$, $\quad b_\lambda = (b - \lambda)/\sqrt{\lambda}$.

Then $P(a \le Y \le b) = P(a_\lambda \le Z_n \le b_\lambda)$ where Z_n is as in part (i), and this is approximated by $P(a_\lambda \le Z \le b_\lambda)$ where Z is the standard Normal random variable.

Graduate Diploma, Module 1, 2009. Question 7

(i) The joint density of X and Y is $f(x, y) = xye^{-(x+y)/2}/16$, for $x > 0$ and $y > 0$.

With $u = x/y$ and $v = y$, we have $x = uv$ and $y = v$. So the partial derivatives are

$$\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u, \quad \frac{\partial y}{\partial u} = 0, \quad \frac{\partial y}{\partial v} = 1,$$

and the Jacobian of the transformation is simply v.

We also have $xy = uv^2$ and $x + y = v(1 + u)$. So the joint density of U and V is

$$\frac{uv^2 e^{-v(1+u)/2}}{16}.v = \frac{uv^3 e^{-v(1+u)/2}}{16}, \quad \text{for } u > 0 \text{ and } v > 0.$$

The pdf of U is $\int_0^\infty \frac{1}{16} uv^3 e^{-v(1+u)/2} dv$ [Put $v(1 + u)/2 = w$]

$$= \int_0^\infty \frac{1}{16} u \frac{8w^3}{(1+u)^3} e^{-w} \frac{2}{1+u} dw$$

$$= \frac{u}{(1+u)^4} \int_0^\infty w^3 e^{-w} dw$$

$$= \frac{u}{(1+u)^4} \times 3! \quad \text{using the result given in the question}$$

$$= \frac{6u}{(1+u)^4}, \quad \text{for } u > 0.$$

(ii) By definition, U being the ratio of two independent χ^2_4 random variables (each of which may be regarded as divided by its number of degrees of freedom), U follows the $F_{4,4}$ distribution.

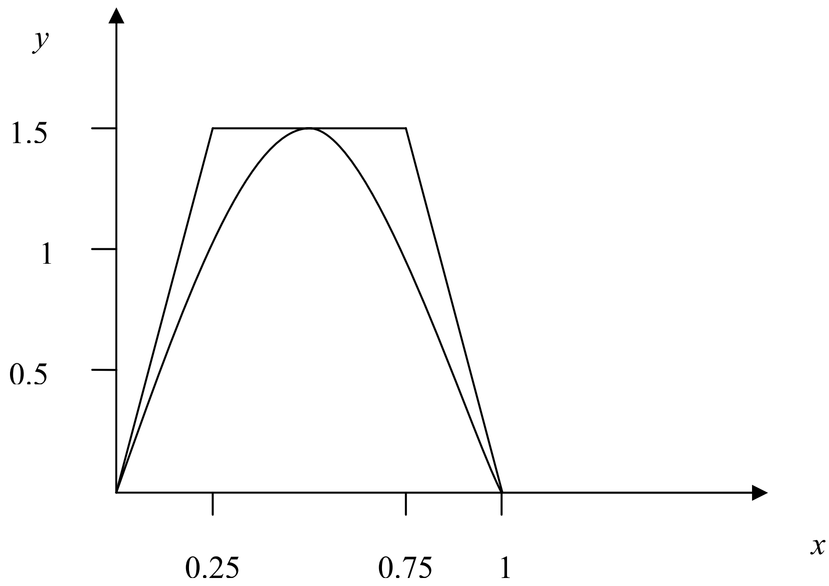
$$E(U) = \int_0^\infty \frac{6u^2}{(1+u)^4} du \quad \text{[Put } t = 1 + u\text{]}$$

$$= \int_0^\infty \frac{6(t-1)^2}{t^4} dt = 6\int_0^\infty \left(t^{-2} - 2t^{-3} + t^{-4}\right) dt = 6\left[1 - 1 + \frac{1}{3}\right] = 2.$$

$U = X/Y$, so we have $E(X/Y) = 2$. As X and Y have the same distribution, $E(X) = E(Y)$ and thus $E(X)/E(Y) = 1$, i.e. less than $E(X/Y)$. There is no mathematical reason to expect that these two quantities should be equal, even if X and Y were independent. Here, the distribution of X/Y (= U) is very skewed to the right, leading to a higher mean than might at first have been supposed.

Graduate Diploma, Module 1, 2009. Question 8

Part (i)



The maximum value of $f(x)$ occurs at $x = 0.5$ and is 1.5, which is the height of the trapezium at this value of x.

$f(x)$ passes through both $x = 0$ and $x = 1$. The derivative is $6 - 12x$, which has value 6 at $x = 0$ and then decreases; 6 is the upwards slope of the trapezium at this point, so $f(x)$ is inside the trapezium there. By symmetry, it is also inside the trapezium at $x = 1$.

Thus it is contained entirely within the trapezium.

Part (ii)

The area under the density function is, of course, 1. By simple geometry, the area of the trapezium is 9/8. Thus the area of the density function occupies 8/9ths of the area of the trapezium and thus, in the long run, 1/9 of the points scattered uniformly over the trapezium will be rejected.

Solution continued on next page

Part (iii)

The method is to reject the point as unsuitable if it does not lie beneath the density function, and to accept the x-value as a suitable point otherwise.

(0.452, 0.697) is plainly under the curve, so accept 0.452.

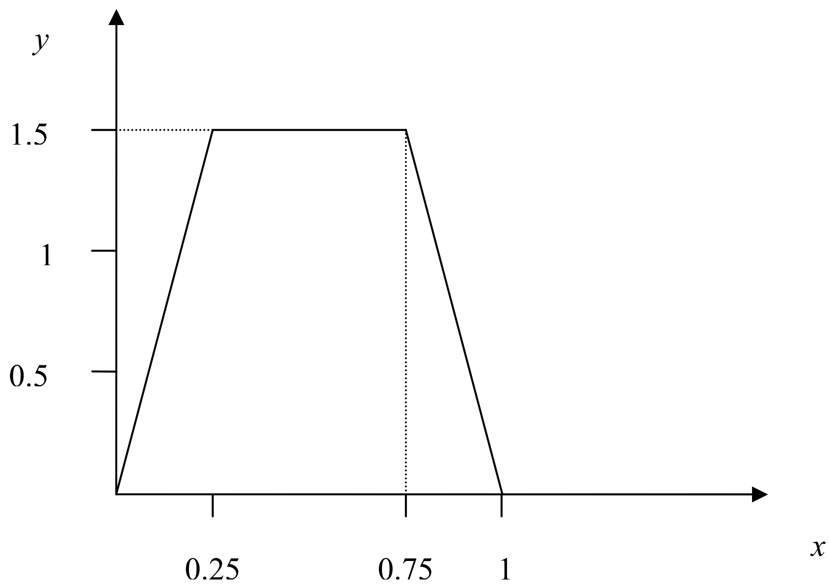
(0.120, 0.066) is also under the curve ($f(0.120) = 0.6336$), so accept 0.120.

(0.914, 0.871) is not under the curve ($f(0.914) = 0.4716$), so reject 0.914.

(0.657, 1.345) is under the curve ($f(0.657) = 1.3521$), so accept 0.657.

Part (iv)

Insert additional lines on the graph as shown, creating two triangles that are clearly congruent.



It follows that the area of the original trapezium is equal to that of the rectangle defined by corner points (0, 0), (0.75, 0), (0.75, 1.5), (0, 1.5). If x and y are two of the stream of random numbers uniformly distributed over the interval (0, 1), the point $w = (3x/4, 3y/2)$ is uniformly distributed over this rectangle. Therefore the method is to generate such points w. Points that lie within the trapezium are used as they stand. Any generated point that lies in the triangle to the left of the trapezium is transformed to the corresponding point in the triangle on the right.