THE ROYAL STATISTICAL SOCIETY

2006 EXAMINATIONS – SOLUTIONS

HIGHER CERTIFICATE

PAPER I – STATISTICAL THEORY

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Higher Certificate, Paper I, 2006. Question 1

(i) The first place can be occupied by 9 different digits, 1 to 9. Each of the other three places can be occupied by 10 digits, 0 to 9.

Hence there are $9 \times 10 \times 10 \times 10 = 9000$ possible PINs.

(ii) All of the combinations in (i) are allowed except 1111, 2222, …, 9999, so there are $9000 - 9 = 8991$ possibilities.

(iii) Only the 9 digits 1 to 9 can be used. The first place can be filled in 9 ways, the second in 8, the third in 7 and the last in 6. So there are $9 \times 8 \times 7 \times 6 = 3024$ possibilities.

(iv) With all 10 digits possible in any position, there would be 10^4 PINs. There are 7 increasing sequences (0123, 1234, …, 6789) and 7 decreasing sequences (9876, 8765, …, 3210), which are not allowed. The number of possible PINs is therefore $10^4 - 14 = 9986$.

(v) All of the 10^4 combinations are allowed except:

(a) the 10 where all 4 digits are the same: 0000, 1111, …, 9999;

(b) those where one digit occurs three times and another just once. There are $10 \times 9 = 90$ ways of choosing the two digits. But note that, for example, 2333, 3233, 3323 and 3332 are four different PINs; whichever two digits occur, the odd one out can be in any of the 4 places in the PIN. Therefore there are $4 \times 90 = 360$ PINs of this sort.

The number of possible PINs is therefore $10^4 - 10 - 360 = 9630$.

Higher Certificate, Paper I, 2006. Question 2

(i) A: (a) $P(0 \text{ entries}) = \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25\,.$

(b) $P(1 \text{ entry}) = 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} = 0.5.$

B: (a) $P(0 \text{ entries}) = \left(\frac{3}{4}\right)^3 = \frac{27}{64} = 0.4219\,.$

(b) $P(1 \text{ entry}) = 3 \times \frac{1}{4} \times \left(\frac{3}{4}\right)^2 = \frac{27}{64} = 0.4219.$

C: (a) $P(0 \text{ entries}) = \left(\frac{4}{5}\right)^5 = \frac{1024}{3125} = 0.3277\,.$

(b) $P(1 \text{ entry}) = 5\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^4 = \frac{256}{625} = 0.4096\,.$

(ii) P(1 entry in total)

= P(1 from A, 0 from B and C) + P(1 from B, 0 from A and C)
+ P(1 from C, 0 from A and B)

$$= \frac{1}{2} \times \frac{27}{64} \times \frac{1024}{3125} + \frac{27}{64} \times \frac{1}{4} \times \frac{1024}{3125} + \frac{256}{625} \times \frac{1}{4} \times \frac{27}{64} = \frac{459}{3125}\,.$$

[If worked in decimals, this is 0.1469.]

P(1 from A | 1 in total) = P(1 from A and 1 in total) / P(1 in total)

= P(1 from A, 0 from B and C) / P(1 in total)

$$= \frac{\frac{1}{2} \times \frac{27}{64} \times \frac{1024}{3125}}{\frac{459}{3125}} = \frac{8}{17}\,.$$

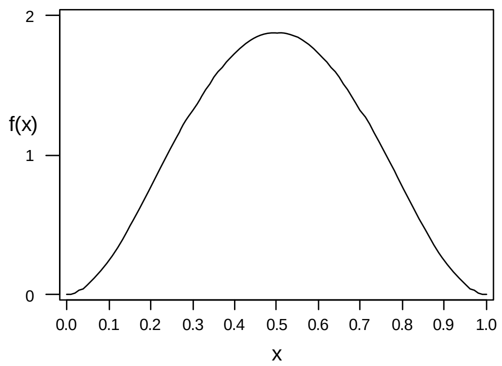
(iii) Denote the numbers of entries from A, B, C as (0, 0, 0) etc. Then we need $P(2, 0, 0) + P(0, 2, 0) + P(0, 0, 2) + P(1, 1, 0) + P(1, 0, 1) + P(0, 1, 1)$. Since entries from each group are independent, we have, as an example, $P(1, 1, 0) =$ P(1 from A).P(1 from B).P(0 from C).

Higher Certificate, Paper I, 2006. Question 3

(i) We have $k\int_0^1 x^2(1-x)^2dx = 1$, so $k\int_0^1\left(x^2-2x^3+x^4\right)dx = 1$. This gives

$$1 = k\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1 = k\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right), \quad \text{so } k = 30.$$

$f(x) = 0$ at $x = 0$ and at $x = 1$. $f(x)$ is symmetrical about $x = ½$. The sketch is as follows.



(ii) $E(X) = \frac{1}{2}$ by symmetry [or by direct integration: $\int_0^1 xf(x)dx$].

$$E(X^2) = 30\int_0^1 x^4(1-x)^2dx = 30\int_0^1\left(x^4-2x^5+x^6\right)dx$$

$$= 30\left[\frac{1}{5}x^5 - \frac{1}{3}x^6 + \frac{1}{7}x^6\right]_0^1 = 30\left(\frac{1}{5}-\frac{1}{3}+\frac{1}{7}\right) = 30\times\frac{1}{105} = \frac{2}{7}.$$

$$\therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{2}{7} - \left(\frac{1}{2}\right)^2 = \frac{1}{28}.$$

$$P\left(X \le \frac{1}{3}\right) = \int_0^{1/3} 30\left(x^2-2x^3+x^4\right)dx = 30\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^{1/3}$$

$$= 30\left(\frac{1}{3^4} - \frac{1}{2}.\frac{1}{3^4} + \frac{1}{5}.\frac{1}{3^5}\right) = \frac{30}{81}\left(1-\frac{1}{2}+\frac{1}{15}\right) = \frac{30}{81}\times\frac{17}{30} = \frac{17}{81} \quad (= 0.2099).$$

(iii) The required probability is $\left(1-\frac{17}{81}\right)^5 = \left(\frac{64}{81}\right)^5 = 0.3079.$

(iv) The variance of $\bar{X}$ for a sample of size 5 is $\frac{\text{Var}(X)}{5} = \frac{1/28}{5} = \frac{1}{140} = 0.00714.$

Higher Certificate, Paper I, 2006. Question 4

Let X represent cycling time without delays: $X \sim N(15, 1)$.

(i) $P(X \le 17) = \Phi\left(\frac{17-15}{1}\right) = \Phi(2) = 0.9772$.

[Φ denotes the cdf of the standard Normal distribution as usual.]

(ii) Adding in the delay times, also Normally distributed [N(0.7, 0.09)], and letting T denote the total time:

(a) $T \sim N(15.7, 1.09)$, so $P(T \le 17) = \Phi\left(\frac{17-15.7}{\sqrt{1.09}}\right) = \Phi(1.245) = 0.8934$;

(b) $T \sim N(16.4, 1.18)$, so $P(T \le 17) = \Phi\left(\frac{17-16.4}{\sqrt{1.18}}\right) = \Phi(0.552) = 0.7096$;

(c) $T \sim N(17.1, 1.27)$, so $P(T \le 17) = \Phi\left(\frac{17-17.1}{\sqrt{1.27}}\right) = \Phi(-0.0887) = 0.4646$.

(iii) The number of delays is distributed as B(3, ½). Hence the situations in (i), (ii)(a), (ii)(b) and (ii)(c) arise with probabilities 1/8, 3/8, 3/8 and 1/8 respectively, so the (unconditional) mean of the total journey time is

$$E(T) = \frac{1}{8}\times 15 + \frac{3}{8}\times 15.7 + \frac{3}{8}\times 16.4 + \frac{1}{8}\times 17.1 = \frac{128.4}{8} = 16.05 \text{ minutes.}$$

(iv) Mean time $\bar{T} \sim N\left(16.05, \frac{1.5025}{10}\right)$.

$$P(\bar{T} \le 17) = \Phi\left(\frac{17-16.05}{\sqrt{0.15025}}\right) = \Phi(2.451) = 0.9929.$$

Higher Certificate, Paper I, 2006. Question 5

(i) $E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$.

$E(X^2) = E\left[X(X-1)+X\right] = E\left[X(X-1)\right] + E[X]$.

$$E\left[X(X-1)\right] = \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} = \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} = \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2 .$$

Hence $E(X^2) = \lambda^2 + \lambda$, and $\text{Var}(X) = E(X^2) - \{E(X)\}^2 = \lambda$.

(ii) $L = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$, and hence $\log L = -n\lambda + \sum_{i=1}^{n} x_i \log\lambda + \text{constant}$.

$\therefore \frac{d\log L}{d\lambda} = -n + \frac{\Sigma x_i}{\lambda}$ which on setting equal to zero gives that the maximum likelihood estimate is $\hat{\lambda} = \frac{\Sigma x_i}{n} = \bar{x}$. [Consideration of $\frac{d^2 \log L}{d\lambda^2}$ confirms that this is a maximum: $\frac{d^2 \log L}{d\lambda^2} = \frac{-\Sigma x_i}{\lambda^2} < 0$.]

(iii) $\text{Var}\left(\hat{\lambda}\right) = \text{Var}\left(\bar{X}\right) = \frac{\text{Var}(X)}{n} = \frac{\lambda}{n}$.

Thus the maximum likelihood estimator of Var($\hat{\lambda}$) is $\frac{\hat{\lambda}}{n}$.

By the central limit theorem, $\hat{\lambda}\ (= \bar{X})$ is approximately Normally distributed with mean λ and variance λ/n. We estimate the variance by $\hat{\lambda}/n$, so that we have $\hat{\lambda} \sim \text{N}\left(\lambda,\ \frac{\hat{\lambda}}{n}\right)$, approximately.

Hence an approximate 95% confidence interval is given by

$$0.95 \approx P\left(-1.96 < \frac{\hat{\lambda}-\lambda}{\hat{\lambda}/\sqrt{n}} < 1.96\right),$$

leading to the interval $\left(\hat{\lambda} - 1.96\sqrt{\frac{\hat{\lambda}}{n}}\ ,\quad \hat{\lambda} + 1.96\sqrt{\frac{\hat{\lambda}}{n}}\right)$.

Solution continued on next page

(iv) For the given sample, we have $n = 12$ and $\Sigma x_i = 48$, leading to $\hat{\lambda} = \bar{x} = 4$. The approximate confidence interval is therefore

$$\left(4 - 1.96\sqrt{\frac{4}{12}} \quad \text{to} \quad 4 + 1.96\sqrt{\frac{4}{12}}\right), \quad \text{i.e. } 2.87 \text{ to } 5.13.$$

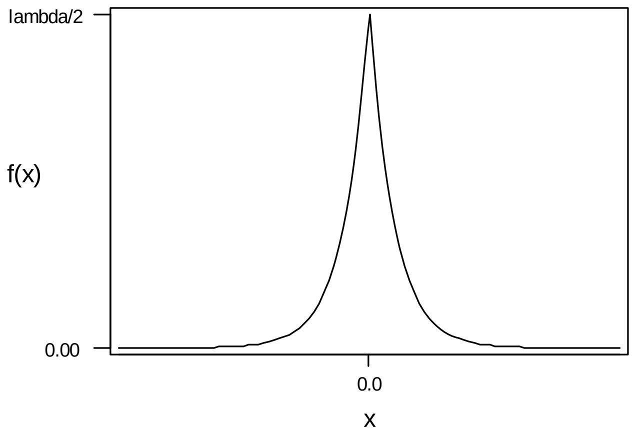
The sample also gives $\Sigma x_i^2 = 238$; so the sample variance is

$$s^2 = \frac{1}{11}\left(238 - \frac{48^2}{12}\right) = \frac{46}{11} = 4.182\,.$$

This is close to the sample mean (4), supporting a Poisson hypothesis for the underlying model.

Higher Certificate, Paper I, 2006. Question 6

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \qquad -\infty < x < \infty$$



By symmetry, $E(X) = 0$.

Hence $\text{Var}(X) = E(X^2) - 0 = \frac{\lambda}{2}\int_{-\infty}^{\infty} x^2 e^{-\lambda|x|} dx = \frac{\lambda}{2}\left\{\int_{-\infty}^{0} x^2 e^{\lambda x} dx + \int_{0}^{\infty} x^2 e^{-\lambda x} dx\right\}$.

Substituting $u = -x$ in the first integral gives $\int_0^\infty u^2 e^{-\lambda u} du$, which is the same as the second. Hence we get, integrating by parts,

$$E(X^2) = \lambda \int_0^\infty x^2 e^{-\lambda x} dx$$

$$= \lambda \left\{ \left[x^2 \frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty + \int_0^\infty \frac{e^{-\lambda x}}{\lambda} .2x\, dx \right\}$$

$$= [0-0] + \int_0^\infty 2x e^{-\lambda x} dx$$

$$= 2\left\{ \left[x \frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty + \int_0^\infty \frac{e^{-\lambda x}}{\lambda} dx \right\}$$

$$= [0-0] + \frac{2}{\lambda}\left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty = \frac{2}{\lambda^2}.$$

Solution continued on next page

If Q, q are the upper and lower quartiles, we have $\int_0^Q \frac{1}{2}\lambda e^{-\lambda x}dx = \frac{1}{4}$, and q will be the same distance below 0 by symmetry.

$\therefore \frac{1}{4} = \left[-\frac{1}{2}e^{-\lambda x}\right]_0^Q = \frac{1}{2}\left(-e^{-\lambda Q}+1\right)$, giving $\frac{1}{2} = 1 - e^{-\lambda Q}$. Therefore $\lambda Q = \log 2$. Hence the semi-interquartile range is $(\log 2)/\lambda$.

$L = \prod_{i=1}^{n}\left(\frac{\lambda}{2}e^{-\lambda|x_i|}\right) = \left(\frac{\lambda}{2}\right)^n e^{-\lambda\sum|x_i|}$, and hence $\log L =$ constant $+ n\log\lambda - \lambda\sum_i |x_i|$.

$\therefore \frac{d\log L}{d\lambda} = \frac{n}{\lambda} - \sum_i |x_i|$ which on setting equal to zero gives that the maximum likelihood estimate is $\hat{\lambda} = \frac{n}{\sum_i |x_i|}$. [Consideration of $\frac{d^2\log L}{d\lambda^2}$ confirms that this is a maximum: $\frac{d^2\log L}{d\lambda^2} = \frac{-n}{\lambda^2} < 0$.]

Higher Certificate, Paper I, 2006. Question 7

(i) The sum of all 12 table entries is $30c$. These probabilities must add up to 1, so $c = 1/30$.

(ii) The marginal distributions are given by the row and column totals.

Hence: $P(X = 1) = 15c = 1/2$; $P(X = 2) = 10c = 1/3$; $P(X = 3) = 5c = 1/6$.

Similarly: $P(Y = 1) = 12c = 2/5$; $P(Y = 2) = 6c = 1/5$; $P(Y = 3) = 6c = 1/5$; $P(Y = 4) = 6c = 1/5$.

(iii) $$E(X) = \left(1\times\frac{1}{2}\right)+\left(2\times\frac{1}{3}\right)+\left(3\times\frac{1}{6}\right) = \frac{1}{2}+\frac{2}{3}+\frac{1}{2} = \frac{5}{3}.$$

$$E(X^2) = \left(1\times\frac{1}{2}\right)+\left(4\times\frac{1}{3}\right)+\left(9\times\frac{1}{6}\right) = \frac{1}{2}+\frac{4}{3}+\frac{3}{2} = \frac{10}{3}.$$

$$\therefore \text{Var}(X) = \frac{10}{3}-\left(\frac{5}{3}\right)^2 = \frac{5}{9}.$$

We also need $E(Y)$ later: $E(Y) = \frac{2}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5} = \frac{11}{5}$.

Distribution of XY:

Values of xy	1	2	3	4	6	12	
Probability	$6c$	$7c$	$4c$	$6c$	$5c$	$2c$	[$c = 1/30$, see above]

$$E(XY) = \left(1\times\frac{6}{30}\right)+\left(2\times\frac{14}{30}\right)+\left(3\times\frac{4}{30}\right)+\left(4\times\frac{6}{30}\right)+\left(6\times\frac{5}{30}\right)+\left(12\times\frac{2}{30}\right) = \frac{110}{30} = \frac{11}{3}$$

Also we have $E(X)E(Y) = \frac{5}{3}\times\frac{11}{5} = \frac{11}{3}$.

$$\therefore \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0.$$

(iv) X and Y are not independent [even though $\text{Cov}(X, Y) = 0$ and even though some cells have $P(X = x, Y = y) = P(X = x).P(Y = y)$]. For example, we have $P(X = 1, Y = 4) = 2/15$, but $P(X = 1).P(Y = 4) = 1/10$.

Solution continued on next page

(v) $U = 1$ if $X = 1$ or 3 $U = 0$ if $X = 2$

$V = 1$ if $Y = 1$ or 3 $V = 0$ if $Y = 2$ or 4

Table of joint distribution of U and V, with margins.

		Values of V		
		0	1	
Values of U	0	$2c = 1/15$	$8c = 4/15$	$10c = 1/3$
	1	$10c = 1/3$	$10c = 1/3$	$20c = 2/3$
		$12c = 2/5$	$18c = 3/5$	

Consider the cell with $(U, V) = (0, 0)$. The cell probability is 1/15 but the product of the marginal probabilities is 2/15. So U and V are not independent.

Higher Certificate, Paper I, 2006. Question 8

(i) $Y_i = a + bx_i + e_i$, $i = 1, 2, \ldots, n$.

The $\{e_i\}$ are uncorrelated random variables with mean 0 and constant variance σ^2.

The method of least squares is equivalent to the method of maximum likelihood for estimating the regression coefficients (a and b) if the $\{e_i\}$ are Normally distributed.

[If the analysis is to proceed to *inference* for the regression coefficients, Normality of the $\{e_i\}$ is required.]

(ii)(a) For $Y_i = \beta x_i + e_i$, we minimise $S = \sum e_i^2 = \sum (y_i - \beta x_i)^2$.

We have $\frac{dS}{d\beta} = -2\sum x_i (y_i - \beta x_i)$ which on setting equal to zero gives $\sum x_i y_i = \beta \sum x_i^2$, so the least squares estimate is $\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$.

[Consideration of $\frac{d^2 S}{d\beta^2}$ confirms that this is a minimum: $\frac{d^2 S}{d\beta^2} = 2\sum x_i^2 > 0$.]

(b) See scatter plot at foot of page. It shows an increasing trend, roughly linear; but there seems to be some increase in variability as x increases. There are not enough data points to be sure.

The usual summary statistics (not all required for the zero intercept model) are

$n = 10$, $\Sigma x_i = 180$, $\Sigma y_i = 40$, $\Sigma x_i^2 = 5150$, $\Sigma y_i^2 = 244$, $\Sigma x_i y_i = 1055$.

$\therefore \hat{\beta} = 1055/5150 = 0.205$. So the fitted line is $y = 0.205x$.

Hence the estimated expected number of violations for $x = 20$ is $0.205 \times 20 = 4.1$.

Logically, zero traffic flow should imply zero speed violations, so that y should be 0 when x is 0, i.e. the zero intercept model seems reasonable. The scatter plot does not contradict this.