EXAMINATIONS OF THE HONG KONG STATISTICAL SOCIETY

GRADUATE DIPLOMA, 2003

Statistical Theory and Methods II

Time Allowed: Three Hours

Candidates should answer **FIVE** *questions.*

All questions carry equal marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use silent, cordless, non-programmable electronic calculators.

Where a calculator is used the ***method*** *of calculation should be stated in full.*

The notation log *denotes logarithm to base* ***e****.*
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^nC_r$.

This examination paper consists of 6 printed pages, **each printed on one side only**.
This front cover is page 1.
Question 1 starts on page 2.

There are 8 questions altogether in the paper.

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with probability density function

$$f(x) = \frac{\theta}{(1+x)^{\theta+1}}, \qquad x > 0,$$

where $\theta > 0$.

(i) Obtain the maximum likelihood estimator of θ and hence find the maximum likelihood estimator, $\hat{\gamma}$, of $\gamma = 1/\theta$.

(6)

(ii) By considering $P(\log(1+X_i) > w)$, show that $\log(1+X_i)$ is exponentially distributed with mean γ. Hence show that $\hat{\gamma}$ is an unbiased estimator of γ.

(6)

(iii) Find the Cramér-Rao lower bound for the variance of unbiased estimators of γ and show that the variance of $\hat{\gamma}$ attains this bound.

(5)

(iv) Is it possible to find an unbiased estimator of θ whose variance attains the corresponding Cramér-Rao lower bound? Justify your answer.

(3)

2. (i) Explain what is meant by a *sufficient statistic*.

(5)

(ii) Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with probability density function

$$f(x) = \exp(\theta - x), \qquad x > \theta,$$

where $-\infty < \theta < \infty$. Let Y denote the smallest order statistic. Show that Y is a sufficient statistic for θ.

(6)

(iii) By evaluating $P(Y > y)$, or otherwise, find the probability density function of Y.

(4)

(iv) A statistic of the form $(Y - c)$, for some constant c, is to be used to estimate θ. Find the value of c that minimises the mean square error of $(Y - c)$.

(5)

3. In a particular set of Bernoulli trials, it is widely believed that the probability of a success is $\theta = \frac{3}{4}$. However, an alternative view is that $\theta = \frac{2}{3}$. In order to test H_0: $\theta = \frac{3}{4}$ against H_1: $\theta = \frac{2}{3}$, n independent trials are to be observed. Let $\hat{\theta}$ denote the proportion of successes in these trials.

(i) Show that the likelihood ratio approach leads to a size α test in which H_0 is rejected in favour of H_1 when $\hat{\theta} < k$ for some suitable k.

(5)

(ii) By applying the central limit theorem, write down the large sample distributions of $\hat{\theta}$ when H_0 is true and when H_1 is true.

(4)

(iii) Hence find an expression for k in terms of n when $\alpha = 0.05$.

(5)

(iv) Find n so that this test has power 0.95.

(6)

4. Components are produced in an industrial process and the numbers of flaws in different components are independent and identically distributed with probability mass function

$$p(x) = \theta(1-\theta)^x, \qquad x = 0, 1, 2, \ldots ,$$

where $0 < \theta < 1$. A random sample of n components is inspected; n_0 components are found to have no flaws, n_1 components have one flaw and the remainder have two or more flaws.

(i) Show that the likelihood is

$$L(\theta) = \theta^{n_0+n_1}(1-\theta)^{2n-2n_0-n_1} .$$

(5)

(ii) Find the maximum likelihood estimator of θ and the sample information in terms of n, n_0 and n_1.

(7)

(iii) Hence calculate an approximate 90% confidence interval for θ when 90 out of 100 components inspected have no flaws, and seven have exactly one flaw.

(8)

5. The strengths in newtons of certain fibre bundles are independent with probability density function

$$f(x)=\frac{2x}{\theta^2}\exp\left(-\frac{x^2}{\theta^2}\right), \qquad x>0,$$

where $\theta > 0$. It is decided to test H_0: $\theta = 2$ against H_1: $\theta = 1$.

(i) Construct a sequential probability ratio test so that the Type I and Type II error probabilities are approximately 0.025 and 0.075 respectively.
(7)

(ii) Let X have the above probability density. Show that

$$E\left(X^2\right)=\theta^2.$$

Hence find the approximate expected sample sizes of the test when $\theta = 1$ and when $\theta = 2$.
(8)

(iii) Perform the above test when the first three strengths are 2.2, 2.5, 1.8 respectively.
(5)

6. (i) Explain what is meant by *conjugate prior distribution* in Bayesian inference. (5)

(ii) Let X_1, X_2, …, X_n be a random sample from a Wald distribution with probability density function

$$f(x|\theta) = x^{-3/2}\theta^{1/2}(2\pi)^{-1/2}\exp\left\{-\tfrac{1}{2}\theta\left(x-2+x^{-1}\right)\right\}, \qquad x>0,$$

where $\theta > 0$. Consider the gamma distribution with positive parameters α and β and probability density function

$$g(y) = k_\alpha y^{\alpha-1}\beta^\alpha \exp(-\beta y), \qquad y>0,$$

where k_α is a suitable constant. Show that this distribution is a conjugate prior for θ.

(5)

(iii) Suppose that the above prior is to be used and that the prior mean and variance are both 20. Explain why $\alpha = 20$ and $\beta = 1$ must be used as parameter values in the prior distribution.

(4)

(iv) Suppose also that n = 80 and $\sum_{i=1}^{80}\left(X_i - 2 + X_i^{-1}\right) = 5.0$. Use a Normal approximation to the gamma distribution to calculate an approximate 90% highest posterior density interval for θ.

(6)

[You may use without proof the result that the mean and variance of a gamma distribution with parameters α and β are α/β and α/β^2 respectively.]

7. Let X_1, X_2, …, X_n be a random sample from a Bernoulli distribution with success probability θ, i.e. $P(X = 1) = \theta$, $P(X = 0) = 1 - \theta$. Suppose that θ has a beta(α, β) prior distribution with probability density function

$$g(\theta) = c\theta^{\alpha-1}(1-\theta)^{\beta-1}, \qquad 0 < \theta < 1,$$

where $\alpha > 0$, $\beta > 0$ and c is the normalising constant.

(i) Using a squared error loss function, show that the Bayes estimator of θ is

$$\frac{\alpha + \sum_{i=1}^{n} X_i}{\alpha + \beta + n} .$$

(8)

(ii) When $\alpha = \beta = \tfrac{1}{2}\sqrt{n}$, find the bias of the Bayes estimator and deduce that its risk is

$$\frac{1}{4\left(1+\sqrt{n}\right)^2} .$$

(8)

(iii) What other property does this estimator have as a consequence of its risk not depending on the value of θ?

(4)

[You may use without proof the result that the mean of a beta(α, β) distribution is $\alpha/(\alpha + \beta)$.]

8. Compare and contrast the parametric and non-parametric approaches in classical statistical inference.

(20)