THE ROYAL STATISTICAL SOCIETY

2002 EXAMINATIONS – SOLUTIONS

GRADUATE DIPLOMA

PAPER II – STATISTICAL THEORY & METHODS

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Users of the solutions should always be aware that in many cases there are valid alternative methods. Also, in the many cases where discussion is called for, there may be other valid points that could be made.

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Graduate Diploma, Statistical Theory & Methods, Paper II, 2002. Question 1

(i) Let X_i denote the number of breakages in the ith chromosome. Then the likelihood function is

$$L(\lambda)=\prod_{i=1}^{33}\frac{e^{-\lambda}}{1-e^{-\lambda}}\cdot\frac{\lambda^{x_i}}{x_i!}=\frac{e^{-33\lambda}}{\left(1-e^{-\lambda}\right)^{33}}\cdot\frac{\lambda^{\sum_{i=1}^{33}x_i}}{\prod_{i=1}^{33}x_i!} \qquad \text{for } \lambda>0.$$

Given that $\sum x_i=\left((11\times1)+(6\times2)+\ldots+(1\times13)\right)=122$,

$$\ln\left[L(\lambda)\right]=l(\lambda)=-33\lambda-33\ln\left(1-e^{-\lambda}\right)+122\ln\lambda-\sum_{i=1}^{33}\ln\left(x_i!\right).$$

So $\dfrac{dl}{d\lambda}=-33-\dfrac{33e^{-\lambda}}{1-e^{-\lambda}}+\dfrac{122}{\lambda}=-33-\dfrac{33}{e^{\lambda}-1}+\dfrac{122}{\lambda}$. With the usual regularity conditions, the maximum likelihood estimate $\hat{\lambda}$ satisfies $\dfrac{dl}{d\lambda}=0$, i.e. $-33-\dfrac{33}{e^{\hat{\lambda}}-1}+\dfrac{122}{\hat{\lambda}}=0$.

(ii) $\dfrac{d^2l}{d\lambda^2}=\dfrac{33e^{\lambda}}{\left(e^{\lambda}-1\right)^2}-\dfrac{122}{\lambda^2}$. An iterative algorithm for finding $\hat{\lambda}$ numerically is given by $\lambda_{n+1}=\lambda_n-\left(\dfrac{dl}{d\lambda}\right)_{\lambda=\lambda_n}\Big/\left(\dfrac{d^2l}{d\lambda^2}\right)_{\lambda=\lambda_n}$. An initial estimate λ_0 could be found by plotting $l(\lambda)$ [or $L(\lambda)$] against λ. Alternatively, it is often satisfactory to use the estimator for a non-truncated Poisson, which here would be $\lambda_0=\dfrac{122}{33}=3.70$.

(iii) Using the given value $\hat{\lambda}=3.6$, $P(X=k)=\dfrac{e^{-3.6}}{1-e^{-3.6}}\dfrac{(3.6)^k}{k!}$.

$$P(X=1)=\frac{3.6e^{-3.6}}{1-e^{-3.6}}=\frac{0.0984}{0.9727}=0.1011. \qquad P(X=2)=\frac{3.6}{2}P(X=1)=0.1820.$$

Similarly, $P(X=3)=0.2184$, $P(X=4)=0.1966$, $P(X=5)=0.1415$. Hence $P(X\geq6)=0.1603$. [Note. These probabilities are accurate to 4 d.p., but there is slight rounding in the expected frequencies below.]

x	1	2	3	4	5	≥ 6	TOTAL
observed	11	6	4	5	0	7	33
expected	3.34	6.01	7.21	6.49	4.67	5.28	33.00

Comparing the observed and expected frequencies, the χ^2 test will have 4 d.f. since λ had to be estimated. The test statistic is

$$X^2=\frac{(11-3.34)^2}{3.34}+\frac{(6-6.01)^2}{6.01}+\frac{(4-7.21)^2}{7.21}+\frac{(5-6.49)^2}{6.49}+\frac{4.67^2}{4.67}+\frac{(7-5.28)^2}{5.28} = 24.57.$$

This is very highly significant as an observation from χ^2_4, i.e. there is very strong evidence against the null hypothesis of a truncated Poisson distribution.

Graduate Diploma, Statistical Theory & Methods, Paper II, 2002. Question 2

(a) Suppose that the data $\underset{\sim}{x} = \left(x_1 \;\; x_2 \;\; \ldots \;\; x_n\right)^T$ have joint probability density (or mass) function $f\left(\underset{\sim}{x}, \theta\right)$, θ being an unknown parameter. The loss $L\left[\delta\left(\underset{\sim}{x}\right), \theta\right]$ of a decision rule is the loss associated with choosing that decision.

The risk of δ is $R_\delta(\theta) = E_{\underset{\sim}{X}|\theta}\left[L\left\{\delta\left(\underset{\sim}{X}\right), \theta\right\}\right]$; and the Bayes risk is $r_\pi(\delta) = E_\pi\left[R_\delta(\theta)\right] = \int R_\delta(\theta)\pi(\theta)\,d\theta$, in which $\pi(\theta)$ is the prior distribution of θ.

A prior distribution which leads to posterior distributions in the same family is called conjugate.

(i) The prior distribution of θ is $\pi(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad 0<\theta<1$.
$X \mid \theta$ is binomial, X being the number of seeds germinating out of n. Hence the posterior distribution of θ is

$$\pi(\theta \mid x) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}.\theta^{x}(1-\theta)^{n-x} = \theta^{\alpha+x-1}(1-\theta)^{\beta+n-x-1}, \quad 0<\theta<1.$$

This is beta with parameters $\alpha+x$ and $\beta+n-x$. Therefore

$$\pi(\theta \mid x) = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)\Gamma(\beta+n-x)}\theta^{\alpha+x-1}(1-\theta)^{\beta+n-x-1}, \quad \text{for } 0<\theta<1.$$

(ii) With a quadratic loss function, the Bayes estimate of θ is equal to the mean of θ under the posterior distribution.

This is $E(\theta \mid x) = \int_0^1 \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)\Gamma(\beta+n-x)}\theta^{\alpha+x}(1-\theta)^{\beta+n-x-1}\,d\theta$

$$= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)}.\frac{\Gamma(\alpha+x+1)}{\Gamma(\alpha+\beta+n+1)} = \frac{\alpha+x}{\alpha+\beta+n}.$$

(iii) If $d = d_0$, the posterior expected loss is

$$cE\left[\theta^2 \mid x\right] = c\int_0^1 \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)\Gamma(\beta+n-x)}\theta^{\alpha+x+1}(1-\theta)^{\beta+n-x-1}\,d\theta$$

$$= c\frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)}.\frac{\Gamma(\alpha+x+2)}{\Gamma(\alpha+\beta+n+2)} = \frac{c(\alpha+x)(\alpha+x+1)}{(\alpha+\beta+n)(\alpha+\beta+n+1)}.$$

Since the loss under $d = d_1$ is 1, choose d_1 if $cE\left[\theta^2 \mid x\right] > 1$,

i.e. if $\dfrac{\alpha+x}{\alpha+\beta+n} > \dfrac{1}{c}.\dfrac{\alpha+\beta+n+1}{\alpha+x+1}$.

A uniform prior has $\alpha = 1$, $\beta = 1$. Hence for $n = 15$, $x = 10$ and $c = 25$, choose d_1 since $\dfrac{11}{17} > \dfrac{1}{25}.\dfrac{18}{12}$ $(0.647 > 0.06)$.

Graduate Diploma, Statistical Theory & Methods, Paper II, 2002. Question 3

(i) Suppose that the data consist of pairs (x_i, y_i) (for i = 1 to n) of observations taken on n units from a population. Let the ranks of the $\{x_i\}$ be $\{v_i\}$ and those of the $\{y_i\}$ be $\{w_i\}$, for i = 1 to n.

Define $d_i = v_i - w_i$ (for i = 1 to n).

Spearman's rank correlation coefficient r_s is the product-moment correlation coefficient of the ranks (v_i, w_i) for i = 1 to n. It may be calculated as

$$r_s = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)}.$$

(ii)

Observation	1	2	3	4	5	6	7
Rank in A	1	2	3	4	5	6	7
Rank in B	1	2	3	4	5	7	6

$$\sum d_i^2 = 2.$$

There are 7! possible rankings altogether. We need to find the number of ways in which a value of $\sum d_i^2 \le 2$ can arise. Keeping the A ranking fixed, the B ranking could be

1 2 3 4 5 6 7	1 3 2 4 5 6 7	1 2 3 5 4 6 7	1 2 3 4 5 7 6
2 1 3 4 5 6 7	1 2 4 3 5 6 7	1 2 3 4 6 5 7	

This is 7 ways out of 7! for the B ranking, i.e. the probability (p-value) is $\frac{7}{7!} = \frac{1}{6!} = \frac{1}{720}$.

(iii)

Environment	1	2	3	4	5	6	7	8
Rank X	2	1	5	7	3	6	4	8
Rank Y	4	5	1	7	3	6	2	8
d_i	–2	–4	4	0	0	0	2	0

$$\sum d_i^2 = 40$$

$$r_s = 1 - \frac{6\times 40}{8\times 63} = 1 - \frac{30}{63} = \frac{33}{63} = \frac{11}{21} = 0.5238.$$

The 5% critical value of r_s for n = 8 is 0.738. Hence there is no evidence of association (at the 5% level).

[Note. The 5% critical value is wrongly quoted in Table XVI in some copies of the Society's Abridged Tables for Examination Candidates as 0.714. Candidates were, of course, not penalised in the examination.]

Graduate Diploma, Statistical Theory & Methods, Paper II, 2002. Question 4

The power of a test is the probability of rejecting the null hypothesis expressed as a function of the parameter under investigation. If both the significance level of the test and the power required at a particular value of the parameter are specified, then a lower bound for the necessary sample size can be determined.

(i) The likelihood function is $L(\theta) = \prod_{i=1}^{n} \theta\lambda x_i^{\lambda-1} e^{-\theta x_i^\lambda} = \theta^n \lambda^n \prod_{i=1}^{n} \left(x_i^{\lambda-1}\right).e^{-\theta\sum_{i=1}^{n} x_i^\lambda}$, for $\theta > 0$, and so the likelihood ratio for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ (where $\theta_0 > \theta_1$) is

$$\Lambda = \frac{L(\theta_0)}{L(\theta_1)} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left\{-(\theta_0 - \theta_1)\sum_{i=1}^{n} x_i^\lambda\right\}.$$

By the Neyman-Pearson lemma, the most powerful test has critical region $c = \left\{\underset{\sim}{x} : \sum_{i=1}^{n} x_i^\lambda \geq k\right\}$, k being chosen to give significance level α for the test.

(ii) $P(X > x) = \int_x^\infty \theta\lambda t^{\lambda-1} e^{-\theta t^\lambda} dt = \left[-e^{-\theta t^\lambda}\right]_x^\infty = e^{-\theta x^\lambda}, \qquad x > 0.$

Hence $P\left(X^\lambda > x\right) = P\left(X > x^{1/\lambda}\right) = e^{-\theta x}, \qquad x > 0.$

So $P\left(X^\lambda \leq x\right) = 1 - e^{-\theta x}, \qquad x > 0, \qquad$ so $X^\lambda \sim \text{Exp}(\theta)$.

(iii) Using the given result, $2\theta\sum_{i=1}^{n} X_i^\lambda \sim \chi^2_{2n}$.

Under H_0 [$\theta = 0.05$], $0.1\sum_{i=1}^{50} X_i^\lambda \sim \chi^2_{100}$, with 1% point 135.81.

Thus $P\left(0.1\sum_{i=1}^{50} X_i^\lambda \geq 135.81 \mid \theta = 0.05\right) = 0.01$, and the test therefore rejects H_0 if $\sum_{i=1}^{50} x_i^\lambda \geq 1358.1$.

(iv) Under H_1 [θ = 0.025], $0.05\sum_{i=1}^{50} X_i^\lambda \sim \chi^2_{100}$, and so the power of the test is

$$P\left(\sum_{i=1}^{50} X_i^\lambda \geq 1358.1 \mid \theta = 0.025\right) = P\left(0.05\sum_{i=1}^{50} X_i^\lambda > 67.905\right) = P\left(\chi^2_{100} > 67.905\right) \approx 0.995.$$

Given a random sample $\underset{\sim}{x} = (x_1 \;\; x_2 \;\; \dots \;\; x_n)^T$ from a distribution whose pdf contains a parameter θ, the likelihood function for this sample is $L(\theta) \equiv f(\underset{\sim}{x}, \theta)$ considered as a function of θ. The maximum likelihood estimator, $\hat{\theta}$, of θ is the value of θ that maximises $L(\theta)$.

For large samples, under standard regularity conditions, $\hat{\theta} \sim$ approx $N\left(\theta, \dfrac{1}{I(\theta)}\right)$, where $I(\theta) = -E\left(\dfrac{d^2 l}{d\theta^2}\right)$ is Fisher's "information function" and $l(\theta) = \ln L(\theta)$.
[$\dfrac{1}{I(\theta)}$ is the Cramér-Rao lower bound for the variance of an unbiased estimator.]

$\hat{\theta}$ is consistent, asymptotically unbiased.

(i) $$E\left[\bar{X}\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n} n\sqrt{\theta} = \sqrt{\theta} \; .$$

$$\text{Var}\left(\bar{X}\right) = \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} n\sqrt{\theta} = \frac{\sqrt{\theta}}{n} \; .$$

$$E\left[\bar{X}^2\right] = \text{Var}\left(\bar{X}\right) + \left(E\left[\bar{X}\right]\right)^2 = \frac{\sqrt{\theta}}{n} + \theta \; .$$

So $E\left[\hat{\theta}\right] = \dfrac{\sqrt{\theta}}{n} + \theta - \dfrac{\sqrt{\theta}}{n} = \theta$, and $\hat{\theta}$ is an unbiased estimator.

(ii) $$L(\theta) = \prod_{i=1}^{n} \frac{e^{-\sqrt{\theta}} \left(\sqrt{\theta}\right)^{x_i}}{x_i!} = \frac{e^{-n\sqrt{\theta}} \left(\sqrt{\theta}\right)^{\sum x_i}}{\prod (x_i!)} ,$$

so $$l(\theta) = \ln L(\theta) = -n\sqrt{\theta} + \sum_{i=1}^{n} x_i \ln\left(\sqrt{\theta}\right) - \ln\left(\prod x_i!\right), \qquad \theta > 0 \; .$$

and $$\frac{dl}{d\theta} = -\frac{n}{2\sqrt{\theta}} + \frac{1}{2\theta} \sum x_i \; .$$

$$\therefore \frac{d^2 l}{d\theta^2} = \frac{n}{4\theta^{3/2}} - \frac{1}{2\theta^2} \sum x_i \quad \text{and} \quad E\left[-\frac{d^2 l}{d\theta^2}\right] = \frac{1}{2\theta^2} E\left[\sum X_i\right] - \frac{n}{4\theta^{3/2}} \; .$$

Thus $I(\theta) = \dfrac{1}{2\theta^2} . n\sqrt{\theta} - \dfrac{n}{4\theta^{3/2}}$ and so the Cramér-Rao lower bound is $\dfrac{4\theta^{3/2}}{n}$.

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Question 5 continued

(iii) When $n = 1$, $\hat{\theta} = X^2 - X$.

$$E\left[\hat{\theta}^2\right] = E\left[\left(X^2 - X\right)^2\right] = \sum_{k=0}^{\infty}\left(k^2 - k\right)\frac{e^{-\lambda}\lambda^k}{k!} = \lambda^2\sum_{k=2}^{\infty}\frac{k(k-1)}{(k-2)!}e^{-\lambda}\lambda^{k-2}$$

$$= \lambda^2\sum_{j=0}^{\infty}(j+1)(j+2)\frac{e^{-\lambda}\lambda^j}{j!} \qquad \text{putting } j = k-2$$

$$= \lambda^2 E\left[(X+1)(X+2)\right].$$

Since $E[X] = \lambda$ and $E\left[X^2\right] = \lambda + \lambda^2$,

$$E\left[\hat{\theta}^2\right] = \lambda^2 E\left[X^2 + 3X + 2\right] = \lambda^2\left(\lambda + \lambda^2 + 3\lambda + 2\right) = \lambda^2\left(\lambda^2 + 4\lambda + 2\right)$$

so that $\text{Var}\left(\hat{\theta}\right) = \lambda^2\left(\lambda^2 + 4\lambda + 2\right) - \lambda^4 = 2\lambda^2(2\lambda + 1)$.

Hence the efficiency of $\hat{\theta}$ is $\dfrac{CRLB}{\text{Var}\left(\hat{\theta}\right)} = \dfrac{4\theta^{3/2}}{n}.\dfrac{1}{2\theta\left(2\sqrt{\theta}+1\right)}$, and since $n = 1$ this is

$\dfrac{1}{1+\dfrac{1}{2\sqrt{\theta}}}$; it $\to 1$ as $\theta \to \infty$.

The opening part of this question is standard bookwork regarding the relationship between statistical tests and confidence sets.

(i) Given $f(x,w)=\dfrac{n(n-1)}{\theta^2}e^{-n(x-\mu)/\theta}e^{-w/\theta}\left(1-e^{-w/\theta}\right)^{n-2}$ [where $x\equiv x_{(1)}$]

for $\mu<x<\infty$ and $0<w<\infty$,

we have $f_W(w)=\dfrac{n(n-1)}{\theta^2}e^{-w/\theta}\left(1-e^{-w/\theta}\right)^{n-2}\int_\mu^\infty e^{-n(x-\mu)/\theta}dx$

$=\dfrac{n(n-1)}{\theta^2}e^{-w/\theta}\left(1-e^{-w/\theta}\right)^{n-2}\int_0^\infty e^{-nv/\theta}dv$ putting $v=(x-\mu)$, so $dv=dx$

$=\dfrac{n(n-1)}{\theta^2}e^{-w/\theta}\left(1-e^{-w/\theta}\right)^{n-2}\left[-\dfrac{\theta}{n}e^{-nv/\theta}\right]_{v=0}^{\infty}$

$=\dfrac{n-1}{\theta}e^{-w/\theta}\left(1-e^{-w/\theta}\right)^{n-2}$.

Therefore $P(W\le w)=\int_0^w\dfrac{n-1}{\theta}e^{-y/\theta}\left(1-e^{-y/\theta}\right)^{n-2}dy$

$=\left[\left(1-e^{-y/\theta}\right)^{n-1}\right]_0^w=\left(1-e^{-w/\theta}\right)^{n-1}$, $0<w<\infty$.

(ii) Let $Z=\dfrac{W}{\theta}$. Then $F_Z(z)=P(Z\le z)=P(W\le z\theta)=\left(1-e^{-z}\right)^{n-1}$, $0<z<\infty$.

Z is a function of θ whose distribution does not depend on θ. Hence it is a pivotal quantity.

(iii) Choose any interval $[z_1,z_2]$, where $z_1\ge 0$, such that

$$\int_{z_1}^{z_2}f_Z(z)\,dz=1-\alpha \qquad \text{for } 0<\alpha<1.$$

Then, given the range $W=w$, we have $z_1\le\dfrac{w}{\theta}\le z_2$, and a $100(1-\alpha)\%$ confidence interval for θ is $\left[\dfrac{w}{z_2},\dfrac{w}{z_1}\right]$.

Graduate Diploma, Statistical Theory & Methods, Paper II, 2002. Question 7

Classical (or "frequentist"). A null hypothesis will specify a model for data, based on a distribution in which there is an unknown parameter; an alternative hypothesis uses the same distribution with different values for the parameter. For example, a null hypothesis can use the model $N(\mu_1, 1)$ with the alternative $N(\mu_2, 1)$. Given the model, a test can be set up with a given probability of rejecting the null hypothesis, for example if a sample mean is "unlikely" to take the value it did in the data, where "unlikely" might mean a probability of less than 0.05. In this case the alternative hypothesis is automatically accepted (even when the null hypothesis is in fact true). The null hypothesis is never "proved", and even with large samples of data there is a measurable chance of making Types I and II errors. It is evidence, not proof, for or against a null hypothesis that is obtained in this method, and misinterpretation is easy in unskilled hands. This remains the most commonly used method of hypothesis testing.

Bayesian. It is unusual to test a simple null hypothesis. But after calculating a confidence interval, a testing process may be carried out by rejecting a null hypothesis that $\theta = \theta_0$ if a $100(1-\alpha)\%$ confidence interval for θ does not contain θ_0. Probabilities can be assigned to opposing hypotheses, and costs can be introduced into this process, much more easily than in others.

Likelihood. If θ_0 does not have a likelihood within a certain distance of the maximum likelihood (i.e. the likelihood for the maximum likelihood estimator $\hat{\theta}$) found from the sample data, the null hypothesis that $\theta = \theta_0$ is rejected. This method depends on using likelihood as a measure of how plausible various values of θ are. The distance from the maximum is sometimes chosen rather arbitrarily.

Graduate Diploma, Statistical Theory & Methods, Paper II, 2002. Question 8

(i) The pdf of X and Y is $f(x,y) = \frac{1}{\theta\phi} e^{-\left(\frac{x}{\theta}+\frac{y}{\phi}\right)}$ for $x, y \ge 0$.

So $P(Y \le X) = \int_0^\infty \int_y^\infty f(x,y)\,dx\,dy$

$$= \int_0^\infty \frac{1}{\phi} e^{-y/\phi} \left[-e^{-x/\theta}\right]_y^\infty dy = \frac{1}{\phi}\int_0^\infty e^{-\left(\frac{\theta+\phi}{\theta\phi}\right)y} dy = \frac{\theta}{\theta+\phi}.$$

(ii) The likelihood function based on n observations of w_i is

$$L_n(\psi) = \left(\sum_{i=1}^{n} w_i\right) \psi^{\sum w_i} (1-\psi)^{n-\sum w_i}, \qquad 0 \le \psi \le 1.$$

The likelihood ratio is $\lambda_n = \frac{L_n(0.5)}{L_n(0.7)} = \left(\frac{0.5}{0.7}\right)^{\sum w_i} \left(\frac{0.5}{0.3}\right)^{n-\sum w_i}$

$$= (0.714)^{\sum w_i} (1.667)^{n-\sum w_i}.$$

The SPR test with the given values of α and β is to

continue sampling while $A < \lambda_n < B$

accept H_0 if $\lambda_n \ge B$

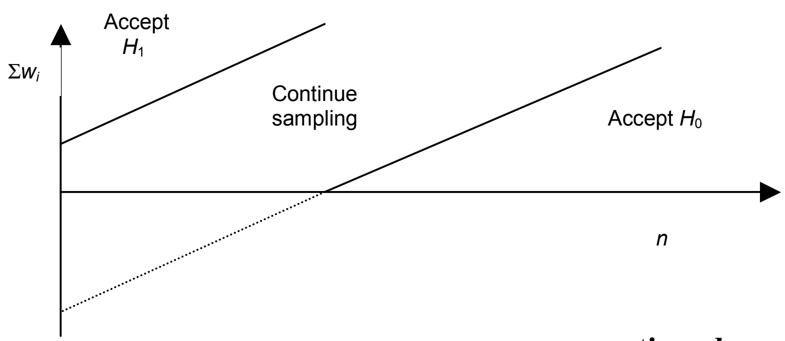
accept H_1 if $\lambda_n \le A$

where $A = \frac{\alpha}{1-\beta} = \frac{0.05}{0.95} = \frac{1}{19}$ and $B = \frac{1-\alpha}{\beta} = \frac{0.95}{0.05} = 19$.

Continue sampling while $\ln A < \sum w_i \ln(0.714) + \left(n - \sum w_i\right)\ln(1.667) < \ln B$

i.e. $0.603n - 3.472 < \sum w_i < 0.603n + 3.472$.

(iii) Plot $\sum w_i$ against n and stop sampling as soon as the sample path crosses one of the boundary lines of the "continue sampling" region.



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Question 8 continued

(iv) Let $z_i = \ln\left(\dfrac{p_0(w_i)}{p_1(w_i)}\right) = w_i \ln(0.714) + (1 - w_i)\ln(1.667)$ for $i = 1, ..., n$.

Then $E_1[Z_i] = 0.7\ln(0.714) + 0.3\ln(1.667) = -0.0825$, and so when H_1 is true the expected sample size is approximately equal to

$$\frac{(1-\beta)\ln A + \beta \ln B}{E_1[Z_i]} = \frac{-0.95\ln 19 + 0.05\ln 19}{-0.0825} = 32.1 .$$