THE ROYAL STATISTICAL SOCIETY

2002 EXAMINATIONS – SOLUTIONS

GRADUATE DIPLOMA

PAPER I – STATISTICAL THEORY & METHODS

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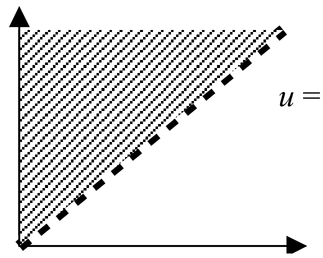
Graduate Diploma, Statistical Theory & Methods, Paper I, 2002. Question 1



(i) $S(t) = \int_t^\infty f(u)\,du \qquad t \ge 0$ and hence

$$\int_0^\infty S(t)\,dt = \int_0^\infty \int_t^\infty f(u)\,du\,dt$$

$$= \int_{u=0}^\infty f(u)\left\{\int_0^u dt\right\} du \text{, integrating over the shaded region,}$$

$$= \int_0^\infty u f(u)\,du = E[T].$$

(ii) $S(x) = \begin{cases} 1 & 0 \le x < 1 \\ x e^{-(x-1)} & x \ge 1 \end{cases}$

$$E[X] = \int_0^\infty S(x)\,dx = \int_0^1 1\,dx + \int_1^\infty x e^{-(x-1)}\,dx$$

$= 1 + \int_0^\infty (u+1) e^{-u}\,du$ putting $u = x - 1$; now use $\Gamma(m)$ result quoted in the question

$= 1 + \Gamma(2) + \Gamma(1) = 1 + 1 + 1 = 3.$

(iii)

$F_Y(y) = \begin{cases} 0 & \text{for } y \le 1 \quad \text{(by definitions of } X \text{ and of } Y\text{)} \\ P(Y \le y) = F_X\left(\sqrt{y}\right) = 1 - \sqrt{y} e^{-\left(\sqrt{y}-1\right)} & \text{for } y > 1 \end{cases}$

$\therefore\ S_Y(y) = \begin{cases} 1 & \text{for } 0 \le y \le 1 \\ \sqrt{y} e^{-\left(\sqrt{y}-1\right)} & \text{for } y \ge 1 \end{cases}$

From (i), $E[Y] = \int_0^1 1\,dy + \int_1^\infty \sqrt{y}\, e^{-\left(\sqrt{y}-1\right)}\,dy$

$= 1 + 2\int_0^\infty (u+1)^2 e^{-u}\,du$ putting $u = \sqrt{y} - 1$

$= 1 + 2\Gamma(3) + 4\Gamma(2) + 2\Gamma(1)$

$= 1 + (2\times 2) + (4\times 1) + (2\times 1) = 11 = E\left[X^2\right].$

Therefore $\text{Var}(X) = E\left[X^2\right] - \{E[X]\}^2 = 11 - 3^2 = 2.$

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[Note that $\int_0^\infty u^{m-1}(1-u)^{n-1}\,du = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$ for all positive integers m, n.]

(a) $$E[U] = \frac{(m+n-1)!}{(m-1)!(n-1)!}\int_0^1 u.u^{m-1}(1-u)^{n-1}\,du$$
$$= \frac{(m+n-1)!}{(m-1)!(n-1)!}\cdot\frac{m!(n-1)!}{(m+n)!} = \frac{m}{m+n}.$$

Similarly, $$E\left[U^2\right] = \frac{(m+n-1)!}{(m-1)!(n-1)!}\cdot\frac{(m+1)!(n-1)!}{(m+n+1)!} = \frac{m(m+1)}{(m+n)(m+n+1)}.$$

$$\therefore \operatorname{Var}(U) = \frac{m(m+1)}{(m+n)(m+n+1)} - \left(\frac{m}{m+n}\right)^2 = \frac{\left(m^2+m\right)(m+n) - m^2(m+n+1)}{(m+n)^2(m+n+1)}$$
$$= \frac{m^3+m^2+m^2n+mn-m^3-m^2n-m^2}{(m+n)^2(m+n+1)} = \frac{mn}{(m+n)^2(m+n+1)}.$$

(b) $$f_X(x) = \int_{y=x}^1 12x^2\,dy = \left[12x^2y\right]_{y=x}^1 = 12x^2(1-x) \qquad \text{(for } 0 \le x \le 1\text{)}.$$
$$f_Y(y) = \int_{x=0}^y 12x^2\,dx = \left[4x^3\right]_{x=0}^y = 4y^3 \qquad \text{(for } 0 \le y \le 1\text{)}.$$

Thus X has beta distribution with m = 3 and n = 2 ["B(3,2)"] and so has mean $\frac{3}{5}$ and variance $\frac{1}{25}$.

Similarly, Y is B(4, 1) and so has mean $\frac{4}{5}$ and variance $\frac{2}{75}$.

$$E[XY] = \int_{y=0}^1\int_{x=0}^y xy.12x^2\,dx\,dy = \int_0^1\left\{\int_0^y 12x^3y\,dx\right\}dy$$
$$= \int_0^1 3y^5\,dy = \left[\frac{1}{2}y^6\right]_0^1 = \frac{1}{2}.$$

$$\therefore \operatorname{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{2} - \frac{3}{5}.\frac{4}{5} = \frac{1}{2} - \frac{12}{25} = \frac{1}{50}.$$

$$\therefore \rho_{XY} = \frac{1}{50}\Big/\sqrt{\frac{1}{25}\times\frac{2}{75}} = \frac{1}{50}\Big/\left(\frac{1}{25}\sqrt{\frac{2}{3}}\right) = \frac{1}{2}\sqrt{\frac{3}{2}} = 0.6124.$$

Graduate Diploma, Statistical Theory & Methods, Paper I, 2002. Question 3

(i) $U^2+V^2=(-2\ln X)(\sin^2 2\pi Y+\cos^2 2\pi Y)=-2\ln X$

$\therefore\ -\frac{1}{2}(U^2+V^2) = \ln X$ so that $X=\exp\left[-\frac{1}{2}(U^2+V^2)\right]$

$\frac{U}{V}=\frac{\sin 2\pi Y}{\cos 2\pi Y}=\tan 2\pi Y$, so $Y=\frac{1}{2\pi}\tan^{-1}\left(\frac{U}{V}\right)$.

(ii) Since X and Y are independent $\mathrm{U}(0,1)$, $f(X,Y)=1$ (for $0\le x\le 1$, $0\le y\le 1$).

The jacobian of the transformation from X, Y to U, V is

$$J=\begin{vmatrix}\frac{\partial X}{\partial U} & \frac{\partial X}{\partial V}\\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V}\end{vmatrix}=\begin{vmatrix}-u\exp\left(-\frac{1}{2}\{u^2+v^2\}\right) & -v\exp\left(-\frac{1}{2}\{u^2+v^2\}\right)\\ \frac{1}{2\pi}.\frac{1}{v}.\frac{1}{1+(u/v)^2} & \frac{1}{2\pi}.\left(-\frac{u}{v^2}\right).\frac{1}{1+(u/v)^2}\end{vmatrix}$$

$$=\frac{1}{2\pi}\exp\left(-\frac{1}{2}\{u^2+v^2\}\right)\left(+\frac{u^2}{v^2}+1\right)\left(\frac{1}{1+(u^2/v^2)}\right) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}\{u^2+v^2\}\right).$$

So $f(u,v) = |J|f(x,y) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(u^2+v^2)\right]$ (for $-\infty<u<\infty$, $-\infty<v<\infty$).

(iii) $f(u,v)$ can be written as the product $\frac{1}{2\pi}g(u)h(v)$, where $g(u)$, $h(v)$ are respectively $\exp\left(-\frac{1}{2}u^2\right)$, $\exp\left(-\frac{1}{2}v^2\right)$. Over $(-\infty,\infty)$, these will integrate to 1 if they have the factor $\frac{1}{\sqrt{2\pi}}$. Hence U and V are independent and both are $\mathrm{N}(0,1)$: $f(u)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}$ and $f(v)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2}$, defined over $(-\infty,\infty)$.

(iv) Generate a pair of uniform random variates x, y in [0, 1], by any suitable process to produce independent variates.

(a) Construct u, v as above to give independent N(0,1) variates.

(b) u^2, v^2 are independent χ_1^2 distributed variates. Hence u^2+v^2 is a χ_2^2 variate.

Graduate Diploma, Statistical Theory & Methods, Paper I, 2002. Question 4

(i) $$M_X(t) = E\left[e^{Xt}\right] = \int_0^\infty e^{xt}.\theta e^{-\theta x}dx = \int_0^\infty \theta e^{-(\theta-t)x}dx$$

$$= \theta\left[\frac{-e^{-(\theta-t)x}}{\theta-t}\right]_0^\infty = \frac{\theta}{\theta-t} \qquad \text{(converges for } t < 0\text{).}$$

$$M'_X(t) = \frac{\theta}{(\theta-t)^2}; \qquad M''_X(t) = \frac{2\theta}{(\theta-t)^3}.$$

$$E[X] = M'_X(0) = \frac{1}{\theta}.$$

$$E\left[X^2\right] = M''_X(0) = \frac{2}{\theta^2}, \quad \text{hence } \operatorname{Var}(X) = \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}.$$

(ii) Using the convolution and "linear transformation" results for moment generating functions,

$$M_Z(t) = e^{-t\sqrt{n}}\left\{M_X\left(\frac{\theta t}{\sqrt{n}}\right)\right\}^n = e^{-t\sqrt{n}}\left(1-\frac{t}{\sqrt{n}}\right)^{-n}$$

$$= e^{-t\sqrt{n}}\left\{1+\left(-\frac{t}{\sqrt{n}}\right)\right\}^{-n},$$

so that

$$\ln M_Z(t) = -t\sqrt{n} - n\ln\left\{1+\left(-\frac{t}{\sqrt{n}}\right)\right\}$$

$$= -t\sqrt{n} - n\left(-\frac{t}{\sqrt{n}} - \frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2 - \frac{1}{3}\left(\frac{t}{\sqrt{n}}\right)^3 - \ldots\right)$$

$$= -t\sqrt{n} + t\sqrt{n} + \frac{1}{2}t^2 + \frac{1}{3}\frac{t^3}{\sqrt{n}} + \ldots$$

$$\to \frac{1}{2}t^2 \text{ as } n \to \infty$$

so that $M_Z(t) \to e^{-\frac{1}{2}t^2}$ as $n \to \infty$.

This is the mgf of $\mathrm{N}(0,1)$, so $Z \to \mathrm{N}(0,1)$.

Graduate Diploma, Statistical Theory & Methods, Paper I, 2002. Question 5

(i) $F_1\left(u_{(1)}\right) = P\left(U_{(1)} \le u_{(1)}\right) = 1 - P\left(U_{(1)} > u_{(1)}\right) = 1 - \left[1 - F\left(u_{(1)}\right)\right]^n$

$= 1 - \left(1 - u_{(1)}\right)^n$ for U(0,1) (for $0 \le u_{(1)} \le 1$).

Hence $f_1\left(u_{(1)}\right) = n\left(1 - u_{(1)}\right)^{n-1}$ (for $0 \le u_{(1)} \le 1$).

(ii) Using the multinomial expression for one observation at u_1, one at u_2 and $n - 2$ observations greater than u_2,

$$f_{1,2}\left(u_{(1)}, u_{(2)}\right) = \frac{n!}{1!1!(n-2)!} 1.1.\left(1 - F\left(u_{(2)}\right)\right)^{n-2} \qquad \text{(since } f\left(u_{(j)}\right) = 1\text{)}$$

$$= n(n-1)\left(1 - u_{(2)}\right)^{(n-2)} \qquad 0 < u_{(1)},\ u_{(2)} < 1.$$

(iii) Change variables to $W = U_{(2)} - U_{(1)}$, $Z = U_{(1)}$.

Hence $U_{(1)} = Z$ and $U_{(2)} = W + Z$.

$$J = \begin{vmatrix} \dfrac{\partial U_{(1)}}{\partial W} & \dfrac{\partial U_{(1)}}{\partial Z} \\ \dfrac{\partial U_{(2)}}{\partial W} & \dfrac{\partial U_{(2)}}{\partial Z} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1, \text{ so } |J| = 1.$$

$\therefore f(w,z) = n(n-1)\left(1 - \{w+z\}\right)^{n-2} \qquad 0 \le w \le 1,\ 0 \le z \le 1,\ 0 \le w+z \le 1.$

$\therefore f_W(w) = n(n-1)\int_{z=0}^{1-w}(1-w-z)^{n-2}dz$ put $z = y(1 - w)$;
then $1 - w - z = (1 - w)(1 - y)$
and $dz = (1 - w)dy$

$$= n(n-1)\int_0^1 \{(1-w)(1-y)\}^{n-2}(1-w)\,dy$$

$$= n(n-1)(1-w)^{n-1}\int_0^1 (1-y)^{n-2}\,dy$$

$$= n(n-1)(1-w)^{n-1}\left[-\frac{(1-y)^{n-1}}{n-1}\right]_0^1 = n(1-w)^{n-1} \qquad \text{(for } 0 \le w \le 1\text{)},$$

which is the same pdf as that of $U_{(1)}$.

(iv) For $n = 10$, $f_W(w) = 10(1-w)^9 \qquad 0 \le w \le 1$.

$$P(W < 0.1) = \int_0^{0.1} 10(1-w)^9\,dw = \left[-(1-w)^{10}\right]_0^{0.1} = 1 - (0.9)^{10} = 0.6513.$$

(i) (a) P(not found) = P(not in region 1) + P(in 1 but not found)
$= \theta_2 + \theta_3 + \theta_1(1-\alpha) = 1 - \alpha\theta_1$.

(b) Let R_i be the event that the aircraft came down in region I and NF the event that it is not found. By Bayes' theorem,

$$P(R_1 \mid NF) = \frac{P(NF \mid R_1)P(R_1)}{P(NF)} = \frac{(1-\alpha)\theta_1}{1-\alpha\theta_1}.$$

At this stage, $P(NF \mid R_2) = P(NF \mid R_3) = 1$ since R_2, R_3 have not been examined.

Hence $P(R_2 \mid NF) = \dfrac{\theta_2}{1-\alpha\theta_1}$ and $P(R_3 \mid NF) = \dfrac{\theta_3}{1-\alpha\theta_1}$.

(ii) Once all three regions have been searched,

$$P(NF) = P(NF \mid R_1)P(R_1) + P(NF \mid R_2)P(R_2) + P(NF \mid R_3)P(R_3)$$
$$= (1-\alpha)\theta_1 + (1-\alpha)\theta_2 + (1-\alpha)\theta_3 = 1-\alpha.$$

So $P(R_i \mid NF) = \dfrac{P(NF \mid R_i)P(R_i)}{(1-\alpha)} = \dfrac{(1-\alpha)\theta_i}{(1-\alpha)} = \theta_i$.

(iii) Given that the aircraft is actually in region i, then it may only be spotted on sorties numbers $3(k-1)+i$, for k = 1, 2, 3, … . The probability that it is spotted for the first time on sortie number $3(k-1)+i$ is $(1-\alpha)^{k-1}\alpha$, since the previous $(k-1)$ sorties in i were "failures".

Hence $E[X \mid \text{aircraft in region } i] = \sum_{k=1}^{\infty}\{3(k-1)+i\}(1-\alpha)^{k-1}\alpha$

$$= 3\alpha\sum_{k=1}^{\infty}k(1-\alpha)^{k-1} + (i-3)\alpha\sum_{k=1}^{\infty}(1-\alpha)^{k-1}.$$

For a geometric series, we have $1 + y + y^2 + y^3 + \ldots\ldots = \dfrac{1}{1-y}$

and $1 + 2y + 3y^2 + \ldots\ldots = \dfrac{d}{dy}\left(\dfrac{1}{1-y}\right) = \dfrac{1}{(1-y)^2}$.

Hence the above sum is $\left(3\alpha.\dfrac{1}{\alpha^2}\right) + \alpha(i-3).\dfrac{1}{\alpha} = \dfrac{3}{\alpha} + i - 3$.

Therefore $E[X] = \left(\dfrac{3}{\alpha} - 2\right)\theta_1 + \left(\dfrac{3}{\alpha} - 1\right)\theta_2 + \left(\dfrac{3}{\alpha}\right)\theta_3 = \dfrac{3}{\alpha} - 2\theta_1 - \theta_2$.

<u>Graduate Diploma, Statistical Theory & Methods, Paper I, 2002. Question 7</u>

(i) First generate by any available method a pseudo-random number between 0 and 1; call it u.

Now set $F(x) = u$, and solve this equation to find $x = F^{-1}(u)$. This value x is a pseudo-random member of the specified distribution.

If this is to work, F must be easily invertible, either algebraically or numerically.

(ii) (a) $F(x) = 1 - e^{-x}$.

If $u = F(x) = 1 - e^{-x}$, then $x = -\ln(1-u)$.

For the given four numbers, using them as u, we find

$x = 0.183;\ 0.269;\ 1.505;\ 3.442.$

[NOTE: if u is U(0,1), so is $(1 - u)$; so $x = -\ln u$ could be used.]

(b) $F(x) = \int_0^x (4t - 4t^3)\,dt = \left[2t^2 - t^4\right]_0^x = 2x^2 - x^4$ (for $0 \le x \le 1$).

If $u = 2x^2 - x^4$, then we have $x^4 - 2x^2 + u = 0$, i.e. $(x^2 - 1)^2 - 1 + u = 0$, or $x^2 - 1 = -\sqrt{1-u}$ (taking negative square root to obtain $x < 1$), which gives $x = \sqrt{1 - \sqrt{1-u}}$. This gives $x = 0.295;\ 0.355;\ 0.727;\ 0.906$.

(c) For the Poisson distribution, tables can be used to set up the cumulative distribution (e.g. Examination Tables XII) or the c.d.f. can be calculated by hand. When $\lambda = 2$, we have:

		u
$P(X=0) = 0.1353$	so $F(0) = 0.1353$	
$P(X=1) = 0.2707$	so $F(1) = 0.4060$	← 0.167, 0.236
$P(X=2) = 0.2707$	so $F(2) = 0.6767$	
$P(X=3) = 0.1804$	so $F(3) = 0.8571$	← 0.778
$P(X=4) = 0.0902$	so $F(4) = 0.9473$	
$P(X=5) = 0.0361$	so $F(5) = 0.9834$	← 0.968

and so on.

Any value of u up to 0.1352 corresponds to $x = 1$; u from 0.1353 to 0.4059 to $x = 2$; and so on. So we find <u>1, 1, 3, 5</u> as the random sample from the Poisson distribution with mean 2.

F needs to be worked out as far into the tail of the distribution as necessary to use all the given values of u.

Graduate Diploma, Statistical Theory & Methods, Paper I, 2002. Question 8

(i) Markov chain model is given by one-step transition matrix:

	L	D	W
L	0.5	0.4	0.1
D	0.3	0.4	0.3
W	0.2	0.4	0.4

Call this $\mathbf{T}$.

(ii) The two-step matrix is

$$\mathbf{T}^2 = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}\begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.39 & 0.40 & 0.21 \\ 0.33 & 0.40 & 0.27 \\ 0.30 & 0.40 & 0.30 \end{pmatrix}$$

So having lost game 1, game 3 is won with probability 0.21.

(iii) $\Pi = (\pi_L \ \pi_D \ \pi_W)$, the stationary distribution, is given by

$\Pi = \Pi\mathbf{T}$, i.e.
$$\pi_L = 0.5\pi_L + 0.3\pi_D + 0.2\pi_W$$
$$\pi_D = 0.4\pi_L + 0.4\pi_D + 0.4\pi_W = 0.4 \quad (\text{using } \pi_L + \pi_D + \pi_W = 1)$$
$$\pi_W = 0.1\pi_L + 0.3\pi_D + 0.4\pi_W$$

So, inserting $\pi_D = 0.4$, we have $0.5\pi_L = 0.12 + 0.2\pi_W$
and $0.6\pi_W = 0.12 + 0.1\pi_L$.

$\therefore\ 3.0\pi_W = 0.60 + 0.5\pi_L = 0.60 + 0.12 + 0.2\pi_W$, i.e. $2.8\pi_W = 0.72$.
Hence $\pi_W = 0.2571$ and $\pi_L = 0.24 + 0.4\pi_W = 0.3429$.

The expected number of points per game is $(0\times\pi_L) + (1\times\pi_D) + (3\times\pi_W) =$ 1.1713.