EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY
(formerly the Examinations of the Institute of Statisticians)

GRADUATE DIPLOMA IN STATISTICS, 1998

Statistical Theory and Methods I

Time Allowed: Three Hours

Candidates should answer **FIVE** *questions.*

All questions carry equal marks.

Graph paper and Official tables are provided.

Candidates may use silent, cordless, non-programmable electronic calculators.

Where a calculator is used the **method** *of calculation should be stated in full.*

Note that $\binom{n}{r}$ *is the same as* ${}^nC_r$ *and that* ln *stands for* $\log_e$.

1. State *Bayes' Theorem*. **(4)**

A car insurance firm believes that 20% of drivers who take out insurance policies with it are 'good' risks, 50% are 'average' risks and 30% are 'bad' risks. In any year, a 'good' risk makes at least one insurance claim with probability 0.05. The corresponding probabilities for 'average' and 'bad' risks are 0.15 and 0.30 respectively.

(i) Find the probability that a policy holder who does not make a claim in a certain year is a 'good' risk. **(4)**

(ii) A random sample of four policy holders is selected from among those who did not make a claim in the previous year. Find the probability that at least two of these policy holders are 'good' risks and at least one more is an 'average' risk. **(7)**

(iii) Find the probability that a policy holder who has not made a claim in the past five years is a 'good' risk. State carefully any simplifying assumptions you make. **(5)**

2. (a) The number of road vehicles arriving at a certain T junction in a period of 5 minutes is a Poisson random variable with expected value m. A proportion q $(0 < q < 1)$ of all the vehicles which arrive at this junction intend to turn left. Derive the (marginal) probability distribution of the number of vehicles which arrive at this junction intending to turn left in a period of 5 minutes. **(10)**

(b) Two motorways, the Mx and the My, merge at a certain point in the roads network. Let the discrete random variables X and Y, respectively, denote the numbers of vehicles that approach this point in a period of five minutes along motorways Mx and My. X follows a Poisson distribution with expected value m $(m > 0)$ independently of Y, which follows a Poisson distribution with expected value n $(n > 0)$. Show that the random variable X + Y follows a Poisson distribution with expected value m + n. **(10)**

3. (a) The continuous random variable U follows a Beta distribution with probability density function

$$f(u) = \frac{(m+n-1)!}{(m-1)!(n-1)!} u^{m-1} (1 - u)^{n-1}, \quad 0 < u < 1$$

where m and n are positive integers. Find the expected value and variance of U in terms of m and n. **(8)**

Question continued on next page

3. (b) The continuous random variables X and Y have joint probability density function

$$f(x, y) = \begin{cases} 6x, & 0 < x < y < 1, \\ 0, & \textit{otherwise} \end{cases}$$

Derive the marginal probability density functions of X and Y, and their expected values and variances. Find the correlation of X and Y.

(12)

4. The continuous random variable X follows a standard Normal distribution. Independently of X, the continuous random variable Y follows the c^2 distribution with probability density function

$$f(y) = \frac{y^{k/2-1}\exp(-y/2)}{2^{k/2}\Gamma(k/2)}, \quad y > 0$$

(where k > 0). Derive the joint probability density function of the random variables

$$U = \frac{X}{\sqrt{Y/k}} \qquad \text{and} \qquad V = \sqrt{Y/k},$$

and find the marginal density function of U. Name the distribution of U and explain briefly why this result is important in Statistics. **(20)**

5. If the continuous random variable Z follows a standard Normal distribution, show that it has moment-generating function

$$M_Z(t) = \exp(½\, t^2).$$

(6)

Suppose that the discrete random variable, X, follows the Poisson distribution with mean m (m > 0). Show that X has moment-generating function

$$M_X(t) = \exp\{(e^t-1)m\}.$$

Hence find the variance of X.

(7)

Also find the moment-generating function of

$$W = \frac{X - \mu}{\sqrt{\mu}}$$

and find its limiting form as $m \to \infty$. [Hint: consider taking the limit of the logarithm of the moment-generating function.] By recognising the limiting moment-generating function, name the limiting distribution of W. **(7)**

6. A random sample of size n is taken from the Exponential distribution with probability density function

$$f(x) = q \exp(-qx), \quad x > 0$$

(where q > 0). The ordered values in the sample are $U_1 \le U_2 \le ... \le U_n$. Derive the joint probability density function of U_1 and U_n.

(6)

Show that the (marginal) probability density function of the range $R = U_n - U_1$ is

$$f(r) = (n-1)\, q \exp(-qr) \{1 - \exp(-qr)\}^{n-2}, \quad r > 0$$

(10)

Find the probability density function of V = exp(-qR).

(4)

7. (a) Let X be any continuous random variable, and let H(x) be its cumulative distribution function. Suppose that U is a continuous random variable which is Uniformly distributed on the interval (0, 1). By considering its distribution function, or otherwise, show that the random variable

$$V = H^{-1}(U)$$

has the same distribution as X, where H^{-1} is the inverse of the distribution function H.

(3)

(b) Using Table IV (Random Digits) in the booklet provided, generate three pseudo-random variates from the Uniform distribution on the interval (0, 1). Work to three decimal places and explain how you found the three variates.

(3)

Use these values to generate three pseudo-random variates from each of the following distributions, and explain how you found them:

(i) Binomial: $P(X = x) = \binom{4}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x}$, x = 0, 1, ..., 4;

(6)

(ii) Uniform: f(x) = ½, -1 < x < 1;

(4)

(iii) standard Normal: $f(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2} x^2)$. [Hint: you may use the result given in part (a).]

(4)

8. The Ehrenfest model of the flow of molecules is based on two urns that contain a total of M balls. At each step of a process, one of the M balls (molecules) is chosen at random and removed from its current urn to the other urn. Let the states of the system be the number of balls (molecules) in one of these urns. Write down the transition probabilities of a Markov Chain model for this process.

(8)

Write down a set of equations that must be satisfied by the stationary distribution of this system, and show that they are satisfied by the following probabilities:

$$P_j = \binom{M}{j}\left(\frac{1}{2}\right)^M, \quad j = 0, 1, ..., M.$$

(12)