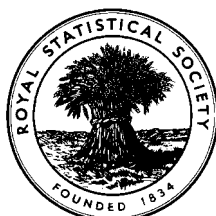


EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY
(formerly the Examinations of the Institute of Statisticians)



GRADUATE DIPLOMA IN STATISTICS, 1997

Statistical Theory and Methods I

Time Allowed: Three Hours

*Candidates should answer **FIVE** questions.*

All questions carry equal marks.

Graph paper and Official tables are provided.

Candidates may use silent, cordless, non-programmable electronic calculators.

*Where a calculator is used the **method** of calculation should be stated in full.*

Note that $\binom{n}{r}$ is the same as nC_r and that \ln stands for \log_e .

1. In a factory, machines A , B and C manufacture 20%, 30% and 50%, respectively, of the total output of switches. Of their respective outputs 3%, 2% and 1% are defective. If a switch drawn at random from the total output is found to be defective, find the probability that it was made by machine A .

An article is made which contains six switches selected at random from the total output, which is large. Find the probabilities that the article

- (i) contains no defective switches,
 - (ii) contains equal numbers of switches from machines A , B and C ,
 - (iii) contains equal numbers of switches from machines A and C .
2. Random variables X and Y have joint probability density

$$f(x, y) = c(y - x)e^{-y}, \quad 0 < x < y.$$

Sketch a graph showing the area where the joint probability density is non-zero.

Show that $c=1$, find the marginal densities of X and Y and identify the distributions of X and Y .

Also find $E(X | Y=y)$ and $E(Y | X=x)$.

3. Show that the moment generating function of a Normal random variable, Z , with zero mean and unit variance is $\exp(t^2/2)$. Find $E(Z^2)$, $E(Z^3)$ and $E(Z^4)$. Deduce the mean and variance of the χ^2 distribution with one degree of freedom.

Let X be a Normal random variable with mean μ and variance σ^2 . Find expressions for $E(X^4)$ and $Var(X^2)$.

4. A group of 20 students consisting of 10 males and 10 females is randomly arranged into 10 pairs. Let $X_i = 1$ if the i th pair consists of a male and a female and let $X_i = 0$ otherwise. Find $E(X_i)$, $Var(X_i)$ and show that $Cov(X_i, X_j) = -10/6137$ for $i \neq j$. Hence, or otherwise, deduce the mean and variance of the number of pairs which consist of a male and a female.

Turn over

5. The joint distribution of X and Y is given by

$$P(X = x, Y = y) = \frac{n! p^x q^y (1-p-q)^{n-x-y}}{x! y! (n-x-y)!}$$

for $x, y = 0, 1, 2, \dots, n$ and $x+y \leq n$, where $p \geq 0, q \geq 0$ and $p+q \leq 1$.

- (i) Find the marginal distribution of X .
 - (ii) Find the conditional distribution of Y given $X=x$ and deduce $E(Y|X = x)$.
 - (iii) A die is thrown 5 times. Let X be the number of times a 6 is thrown and let Y be the number of times a 1 or 2 is thrown. Calculate $P(X=1, Y=2)$ and $P(Y=2|X=1)$.
6. Using a Poisson approximation, find the probability of accepting a large batch in which the proportion of defectives is p , for each of the following sampling schemes.
- (i) Take a random sample of size 100 and accept the batch if the sample contains fewer than 3 defectives, otherwise reject it.
 - (ii) Take a random sample of size 40. Accept the batch if it contains no defectives, reject the batch if it contains more than 2 defectives, otherwise take a second sample of size 100 and accept the batch only if the combined samples contain fewer than 4 defectives.

Determine the expected sample size for scheme (ii) in terms of p . Show that for any p ($0 < p < 1$), the expected sample size is less than that for scheme (i).

7. Give the χ^2 distribution which corresponds to an exponential distribution, mean 2. Describe briefly how the exponential and Poisson distributions are related.

Show how you would use a random sample u_1, u_2, \dots from the uniform distribution on the interval $(0, 1)$ to generate random variables from the following distributions and explain the basis of the method you use.

- (i) Exponential: $P(X \leq x) = 1 - \exp(-x/2)$.
- (ii) χ^2 on 4 degrees of freedom.
- (iii) F on 2 and 2 degrees of freedom.
- (iv) Poisson: $P(Y=k) = \frac{2^k e^{-2}}{k!}$ for $k = 0, 1, 2, \dots$

8. At the end of each hour the number of cars in a car park with limited capacity k (>2) is a Markov chain with state space $\{0, 1, 2, \dots, k\}$ and transition matrix $P = \{p_{ij}\}$, where $p_{00} = p_{01} = p_{02} = 1/3$, $p_{ii-1} = p_{ii} = p_{ii+1} = 1/3$ for $i=1, 2, \dots, k-1$ and $p_{k,k-1} = 1/3$, $p_{k,k} = 2/3$. Find the stationary distribution when $k = 4$.

Given that, for fixed k , the stationary probabilities $\pi_2, \pi_3, \dots, \pi_k$ are all equal, find the stationary distribution for general k and find its mean.